NUMERICAL OPTIMIZATION

Final Technical Report

by

Francesco Zirilli

December 1992

United States Army

EUROPEAN RESEARCH OFFICE OF THE U.S. ARMY

London England

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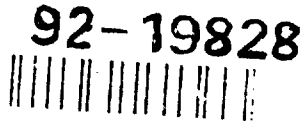
Università di Roma "La Sapienza", Italy



Abstract

In the framework of the proposed "continuous approach" to constrained optimization problems, we describe two new solution methods which resulted from the research. The first is a continuous "inexact" method for solving systems of nonlinear equations and complementarity problems (along the lines of the DAFNE Method), and the second is a continuous method for solving the linear programming problems (along the lines of Karmarkar's method) which is shown to be quadratically convergent.

Some numerical experience on a number of test problems is reported.

Keywords

Numerical Optimization
Constrained Optimization
Systems of nonlinear equations
Complementarity Problems
Linear Programming
Karmarkar's method

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1. Introduction

This is the final report on the work performed from September 1986 to December 1991, under contract n. DAJA 45-86-C-0028 awarded to the University of Rome "La Sapienza" on the research project "Numerical Optimization", by the principal investigator Francesco Zirilli and his co-workers.

The objective of the research is described in par. 2, the results of the research are described in par. 3, and some conclusions are in par. 4.

2. Objective of the research

The subject of the research was the field of those problems in constrained optimization which, starting from the linear programming problem, can be formulated, with growing degree of generalization, first as linear complementarity problems and second as nonlinear complementarity problems.

The objective of the research was to attack the above problems by means of the so-called "continuous approach" to optimization (as opposed to the so-called "pivotal" methods, such as the simplex method for linear programming), with special consideration for the interesting cases of non-convex or ill-conditioned problems, and problems with a very large number of variables; and in particular the objective was to investigate the possibility of applying to the above problems, suitably transformed into nonlinear equation problems, the methods developed by the principal investigator and his co-workers for solving nonlinear equations and global optimization problems, based on the numerical integration of suitable ordinary or stochastic differential equations (refs. [1] to [4]).

3. Results of the research

During the development of the research the complementarity pro-

blems proved to be much more difficult than it had been anticipated and it became clear that the original plans where somehow too ambitious.

The final outcome of the research, if judged against the original plans, is therefore admittedly less satisfactory than it was originally hoped; nevertheless a number of interesting results have been obtained, so that we feel that the research is still to be considered at least partially successful.

The main results of the research are contained in the two papers numbered [5] and [6] in the list of references, which are described in the following paragraphs 3.1 and 3.2, and enclosed as Appendix 1 and Appendix 2.

Report on some work performed in other directions along the lines of the original research plan, together with some related results of auxiliary and preliminary nature, were described in the Periodic Technical Reports; see also the papers numbered [8] and [9] in the list of references.

The research has also stimulated scientific contacts with several italian and foreign scholars.

The above results have been disseminated by means of the aforementioned papers on high-standard academic journals, and seminars at Accademia dei Lincei, Rome (ref. [7], which originated paper [8]), at two meetings of CECAM, Centre Européen de Calcul Atomique et Moléculaire, the first in Ermelo (The Netherlands), ref. [10], and the second at CECAM main office in Orsay (Paris), France, ref. [11].

3.1. The first paper

The first paper (ref. [5], and Appendix 1) can be summarized as follows.

A class of algorithms is developed for the numerical solution of nonlinear systems of equations and complementarity problems, based on the fact that the solution of complementarity problems can be reduced to the solution of systems of nonlinear equations by means of a transformation first suggested by Mangasarian.

The method is "continuous" since it looks for the solution of the non-

linear system by following the numerical solution trajectories of a suitable differential equation, as in previous work of the same authors such as the method implemented in the package DAFNE, described in Refs. [1] and [2].

At each numerical integration step, the DAFNE method requires the solution of an NxN system of linear equations, and the cost of solving such a system when a large numer N of unknowns is involved is the most important part of the computation.

The present method can be called "inexact", since it computes only an approximate solution of the above linear system, by means of a conjugate-gradient procedure which is suitably stopped before "convergence", i.e. after a number $m(\leq N)$ of steps depending on the norm of the residual. For these algorithms local convergence and Q-superlinear rate of convergence has been proved. The algorithms have been used to solve three complementarity problems derived from variational inequalities of mathematical physics very successfully. The complementarity problems considered had up to 900 variables.

3.2. The second paper

The second paper (ref. [6] and Appendix 2) can be summarized as follows.

The paper introduces a new method for solving the linear programming problem, i.e. the problem of minimizing a linear cost function of several real variables, subject to linear equality and inequality constraints.

Following Karmarkar [12], the paper considers the problem in the "canonical" form

minimize $f(\underline{x}) = \underline{c}^T\underline{x}$

subject to

$$A\underline{x} = \underline{0}$$
$$\underline{e}^T\underline{x} = 1$$
$$x_i \geq 0, \quad i=1, \ldots, n$$

where

$$\underline{x} = (x_1,...,x_n)^T,$$
$$\underline{c} = (c_1,...,c_n)^T,$$
$$\underline{e} = (e_1,...,e_n)^T = (1,1,...,1)^T$$

are real column vectors with n elements, A is a real m x n matrix of rank m, with $A\underline{e} = \underline{0}$, $n \geq 2$, $m < n$, and without loss of generality the objective function $f(\underline{x})$ may be "normalized", i.e. $f(\underline{x}^*) = 0$ if $\underline{x}^*$ is a solution of the problem.

In this paper it is shown that Karmarkar's method [12] is in fact equivalent to applying, to a suitable initial value problem for a system of ordinary differential equations, the numerical integration method known as Euler's method with variable stepsize, and obtaining the problem solution $\underline{x}^*$ as the limit, as t goes to infinity, of the numerically computed solution $\underline{x}(t)$ to the initial value problem, starting from the initial point $\underline{x}_0 = (1/n)\underline{e}$.

The proposed method is also based on the above interpretation of Karmarkar's method, but with two main differences:

1) the initial value problem is based on a different system of ordinary differential equations;
2) the numerical integration method is a linearly implicit A-stable method with variable stepsize.

The resulting algorithm is shown to be quadratically convergent.

The computational cost of one step of the proposed algorithm is shown to be of the same order of one step of Karmarkar's algorithm.

While one step of the classical simplex algorithm [13] for linear programming is much cheaper, it may be expected that - due to the quadratic convergence - the number of iterations needed to solve a linear programming problem to a given accuracy, should be approximately independent of the problem size n.

Some numerical results are also reported, which appear to support such expectation.

The algorithm was tested on ten test problems, one originating from

the operations of an industrial plant in central Italy, and the other nine provided by the System Optimization Laboratory at Stanford University.

The results are reported in Table 1 where n is the number of variables, m the number of constraints, k is the index of the first step that verifies the stopping rule

$$f(\underline{x}) \leq 10^{-8} \cdot f(\underline{x}_0)$$

and ν_k is the corresponding value of $f(\underline{x})$.

We note that the test problems with $n,m \leq 5$ are solved in about ten steps, and that, while n and m vary by an order of magnitude, the number k of steps needed to solve the problem varies only by a factor of two.

TABLE 1

Test problem	m	n	k	ν_k
1. ZIR1	304	543	21	2.41D-10
2. ADLITTLE	57	141	21	3.16D-09
3. AFIRO	28	54	12	1.52D-12
4. BEACONFD	173	298	20	3.91D-09
5. BLEND	75	117	21	1.47D-12
6. ISRAEL	175	319	17	1.94D-10
7. SC105	106	166	13	1.36D-11
8. SC50A	51	81	14	1.48D-14
9. SC50B	51	81	11	7.84D-10
10. SHARE2B	97	167	21	1.78D-10

4. Conclusions

The research resulted in two new methods, one for solving complementarity problems, and the other for solving the linear programming problems, both based on the so-called "continuous" approach to optimization.

Successful solution was obtained for the complementarity problems on test problems with up to 900 variables.

However, due to the great difficulty of complementarity problems - in fact much greater than expected - the hoped for attack of much more difficult problems proved to be unsuccessful.

The linear programming method was shown to be quadratically convergent, and was successfully tested on preliminary test problems with up to about 300 variables and 540 constraints.

5. References

[1] F. Aluffi-Pentini, V. Parisi, F. Zirilli
A Differential-Equations Algorithm for Nonlinear Equations,
ACM Trans. on Math. Softw., Vol. 10, n. 3, Sept. 1984, p. 292-316.

[2] F. Aluffi-Pentini, V. Parisi, F. Zirilli
Algorithm 617. DAFNE: A Differential Equations Algorithm for Nonlinear Equations,
ACM Trans. on Math. Softw., Vol. 10, n. 3, Sept. 1984, p. 317-324.

[3] F. Aluffi-Pentini, V. Parisi, F. Zirilli
A Global Optimization Algorithm Using Stochastic Differential Equations, ACM Trans. on Math. Softw., Vol. 14, n. 4, Dec. 1988, p. 345-365.

[4] F. Aluffi-Pentini, V. Parisi, F. Zirilli
Algorithm 667. SIGMA: A Stochastic-Integration Global Minimization Algorithm,
ACM Trans. on Math. Softw., vol. 14, n. 4, Dec. 1988, p. 366-380.

[5] F. Aluffi-Pentini, V. Parisi, F. Zirilli
An Inexact Continuous Method for the Solution of Large Systems of Equations and Complementarily Problems,
Rend. di Mat., Sez. VII, vol. 9, p. 521-543 (1989).

[6] S. Herzel, M.L. Recchioni, F. Zirilli
A Quadratically Convergent Method for Linear Programming,
Lin. Alg. and its Applications, 155, 255-289 (1981).

[7] F. Zirilli
A parallel global optimization algorithm inspired by quantum physics.
Invited talk at the Internat. Sympos. on Vector and Parallel Processors for Scientific Computation, held by the Accademia Nazionale dei Lincei and IBM, Rome, Sept. 1987.

[8] F. Aluffi-Pentini, V. Parisi, F. Zirilli
A parallel global optimization algorithm inspired by quantum physics,
Calcolo, Vol. 25, n. 1-2, Jan.-Jun. 1988.

[9] L. Misici, F. Zirilli
The Inverse Gravimetric Problem: an Application to the Northern San Francisco Craton Granite,
submitted to J. of the Geological Society, London.

[10] F. Zirilli
Some differential equations methods in linear programming,
Workshop: The global optimization problem: techniques and selected applications, CECAM - Centre Européen de Calcul Atomique et Moléculaire, Ermelo, Olanda, 21-24 Aug. 1989.

[11] F. Zirilli
Some physical ideas leading to global optimization algorithms
Workshop: Global optimization in macromolecular chemistry, CE-CAM - Centre Européen de Calcul Atomique et Moléculaire, Orsay (Paris), Francia, 18-29 June 1990.

[12] N. Karmarkar
A New Polynomial Time Algorithm in Linear Programming,
Combinatorica, 4, 373-395 (1984).

[13] G.B. Dantzig
Linear Programming and Extensions,
Princeton University Press, Princeton, N.J., 1963.

APPENDIX 1

Rendiconti di Matematica, Serie VII
Volume 9 , Roma (1989), 521-543

An Inexact Continuous Method for the Solution of Large Systems of Equations and Complementarity Problems

F. ALUFFI-PENTINI - V. PARISI - F. ZIRILLI(*)

Dedicato alla memoria di Carlo Cattaneo, maestro ed amico

RIASSUNTO – *Si considera un nuovo metodo per la risoluzione numerica sia di sistemi di equazioni non lineari sia di problemi di complementarità, che si basa sul fatto che la risoluzione di problemi di complementarità si può ricondurre alla risoluzione di sistemi di equazioni non lineari mediante una trasformazione suggerita da Mangasarian. Il metodo è "continuo" in quanto la soluzione del sistema viene cercata seguendo le traiettorie ottenute per integrazione numerica di un'opportuna equazione differenziale — come in precedenti lavori degli autori — e si può dire "inesatto" nel senso che fa uso di un metodo di gradienti coniugati, opportunamente arrestato "prima della convergenza", per la risoluzione del sistema lineare che nasce nell'integrazione numerica dell'equazione differenziale. Il metodo appare particolarmente efficiente per problemi in cui compare un gran numero di variabili indipendenti, nei quali la parte prevalente dello sforzo di calcolo è rappresentata dalla soluzione di un sistema lineare ad ogni passo di integrazione. Vengono dimostrate la convergenza locale e la convergenza Q-superlineare del metodo, e vengono presentati alcuni risultati numerici relativi a problemi di complementarità della fisica matematica.*

ABSTRACT – *We consider a new method for the numerical solution both of nonlinear systems of equations and of complementarity problems, based on the fact that*

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the solution of complementarity problems can be reduced to the solution of nonlinear systems of equations by means of a transformation first suggested by Mangasarian. The method is "continuous" since it looks for a solution of the nonlinear system by following the numerical solution trajectories of a suitable differential equation — as in previous work by the present authors — and can be called "inexact" since it uses a conjugate-gradient method which is suitably stopped "before convergence" for the solution of the linear system arising in the numerical integration of the differential equation. The method appears to be particularly effective for problems involving a large number of independent variables, where the computational cost is dominated by the solution of a linear system at each integration step. Local convergence and Q-superlinear convergence of the method are proved, under suitable assumptions, and some numerical experience on complementarity problems of mathematical physics is presented.

KEY WORDS – *Numerical analysis - Nonlinear equations - Mathematical programming - Complementarity problems.*

A.M.S. CLASSIFICATION: 65H10 - 65K05

1 – Introduction

Let $\mathbb{R}^N$ be the N-dimensional real euclidean space, let $\mathbf{x} = (x_1, x_2, \ldots, x_N)^T \in \mathbb{R}^N$ be a vector, and for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ let $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{N} x_i y_i$, $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ be the euclidean scalar product and norm; where necessary $\|\cdot\|$ will indicate also the matrix norm induced by the euclidean vector norm. Given $\mathbf{f} : \mathbb{R}^N \to \mathbb{R}^N$ we will be concerned with two classes of problems in this paper: the problem of solving the system of simultaneous nonlinear equations

$$\mathbf{f}(\mathbf{x}) = 0 \tag{1.1}$$

that is: find $\mathbf{x}^* \in \mathbb{R}^N$ such that $\mathbf{f}(\mathbf{x}^*) = 0$, and the complementarity problem

$$\mathbf{x} \geq 0 \tag{1.2}$$

$$\mathbf{f}(\mathbf{x}) \geq 0 \tag{1.3}$$

$$\langle \mathbf{x}, \mathbf{f}(\mathbf{x}) \rangle = 0 \tag{1.4}$$

where $\mathbf{x} \geq 0$ means $x_i \geq 0$, $i = 1, 2, \ldots, N$, and similarly $\mathbf{f}(\mathbf{x}) \geq 0$ means $f_i(\mathbf{x}) \geq 0$, $i = 1, 2, \ldots, N$, $f_i(\mathbf{x})$ being the components of $\mathbf{f}$, that is: find $\mathbf{x}^*$ such that: $\mathbf{x}^* \geq 0$, $\mathbf{f}(\mathbf{x}^*) \geq 0$, $\langle \mathbf{x}^*, \mathbf{f}(\mathbf{x}^*) \rangle = 0$.

The importance of the problem of solving a system of simultaneous equations is well known. When $\mathbf{f}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ is an affine map the (linear) complementarity problem has been considered by COTTLE and DANTZIG in [1] and contains as special cases the linear programming and the quadratic programming problem. In the case when $\mathbf{f}(\mathbf{x})$ is a possibly nonlinear function of $\mathbf{x}$ the (nonlinear) complementarity problem is a rather general problem and contains as special cases the Kuhn-Tucker first-order necessary conditions for the nonlinear programming problem and has been widely studied; see for example GOULD and TOLLE [2].

The linear and nonlinear complementarity problems have applications in such diverse areas of flow in porous media [3], image reconstruction [4], [5], game theory [6].

In this paper we will be concerned with the problem of the numerical solution of nonlinear systems of equations and complementarity problems. Usually complementarity problems are approached numerically with pivotal methods (for example the simplex method for linear programming). The pivotal methods are usually of the "step by step" improvement type, that is, given a problem for which a solution is sought, the standard approach is to attempt to define recursively a sequence of approximate solutions which have the basic property of making an improvement in a suitable "objective function". When the problem satisfies some convexity and/or monotonicity assumptions the pivotal methods are guaranteed to converge and if only a moderate number of independent variable is involved (up to few hundreds) their numerical performance is satisfactory.

In recent years there has been a growing interest in the use of continuous methods in nonlinear optimization; see for example ALLGOWER and GEORG [7] for a review of simplicial methods in the computation of fixed points and the solution of nonlinear equations, and BAYER and LAGARIAS [8] for the interpretation of Karmarkar's linear programming algorithm as a method that follows a trajectory of a suitable system of ordinary differential equations. In particular the present authors have developed a method for solving systems of nonlinear equations based on the numerical integration of an initial-value problem for a system of ordinary differential equations inspired by classical mechanics [9], [10], [11], [12] and a method for global optimization based on the numerical integration of an initial value problem for a system of stochastic differential equations inspired by statistical mechanics [13], [14], [15]. In section 2 the

algorithms introduced in [10] to solve systems of nonlinear equations are modified to obtain an "inexact" solution of the linear systems appearing in each iteration in the spirit of DEMBO, EISENSTAT and STEIHAUG [16]. These new algorithms are particularly effective for problems involving a large number of independent variables where the computational cost is dominated by the solution of the linear system at each step. Under suitable hypotheses local convergence and Q-superlinear convergence of these new "inexact" algorithm for nonlinear systems of equations are proved. In section 3 the complementarity problem is transformed into a nonlinear system of equations following MANGASARIAN [17] and therefore the algorithms previously developed provide a class of locally convergent Q-superlinear methods, which are not of the "step-by-step improvement" type, for the solution of complementarity problems. Finally in section 4 some numerical experience obtained with the algorithms of section 2 and 3 on some complementarity problems of mathematical physics is shown.

Some of the results of this paper have been announced in [18].

2 – Some inexact algorithms for nonlinear systems of equations

Let $\mathbf{f}(\mathbf{x})=(f_1(\mathbf{x}),f_2(\mathbf{x}),\dots,f_N(\mathbf{x}))^T \in \mathbb{R}^N$, where $f_i(\mathbf{x}), i=1,2,\dots,N$, are real-valued regular functions defined for $\mathbf{x}=(x_1,x_2,\dots,x_N)^T \in \mathbb{R}^N$.

In order to solve the system of simultaneous equations

$$\mathbf{f}(\mathbf{x}) = 0 \tag{2.1}$$

we define

$$F(\mathbf{x}) = \mathbf{f}(\mathbf{x})^T\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{N} f_i^2(\mathbf{x}). \tag{2.2}$$

It is easy to see that $\mathbf{x}^*$ is an isolated minimizer of $F(\mathbf{x})$ and $F(\mathbf{x}^*) = 0$.

In [9], [10], [11], [12] the idea has been proposed and developed of associating to the nonlinear system (2.1) the following system of second-order ordinary differential equations:

$$\mu\frac{d^2\mathbf{x}}{dt^2}(t) = -gD\frac{d\mathbf{x}}{dt}(t) - \nabla F(\mathbf{x}(t)) \qquad t \in [0,+\infty) \tag{2.3}$$

Where D is a $N \times N$ positive symmetric matrix, μ, g are positive constants, $\nabla F(\mathbf{x})$ is the gradient of the function $F(\mathbf{x})$ with respect to $\mathbf{x}$. The equation (2.3) represents Newton's second law (mass $\times$ acceleration = force) for a particle of mass μ moving in $\mathbb{R}^N$ subject to the force $-\nabla F$ given by the potential F and to the dissipative force $-gD\dot{\mathbf{x}}/dt$.

If $\mathbf{x}^*$ is an isolated minimizer of $F(\mathbf{x})$ then $\mathbf{x}(t) = \mathbf{x}^*, \forall\, t \in [0, +\infty)$, is a solution of (2.3); consider the Cauchy data

$$\mathbf{x}(0) = \xi_0 \tag{2.4}$$

$$\frac{d\mathbf{x}}{dt}(0) = \eta_0 \tag{2.5}$$

and let $\mathbf{x}(t, \xi_0, \eta_0)$ be the solution of the Cauchy problem (2.3), (2.4), (2.5).

It can be shown that there exists a neighborhood $U \subset \mathbb{R}^{2N}$ of $\begin{bmatrix} \mathbf{x}^* \\ 0 \end{bmatrix} \in \mathbb{R}^{2N}$ such that if $\begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} \in U$ we have:

$$\lim_{t \to \infty} \|\mathbf{x}(t, \xi_0, \eta_0) - \mathbf{x}^*\| = 0 \tag{2.6}$$

Hence in order to solve the system of nonlinear simultaneous equations by integrating numerically the Cauchy problem (2.3), (2.4), (2.5), we are primarily interested in the equilibrium points reached asymptotically by the trajectories of (2.3) (hopefully solutions of (2.1)) rather than in the accuracy of the numerical scheme. So that of particular interest are numerical methods enjoying a special stability property called A-stability [10].

Let $t \in \mathbb{R}$, let $\mathbf{y}, \xi_0 \in \mathbb{R}^m$ and $\varphi(t, \mathbf{y}) \in \mathbb{R}^m$ be a given function continuous in t and continuously differentiable with respect to $\mathbf{y}$, such that the initial-value problem:

$$\frac{d\mathbf{y}}{dt}(t) = \varphi(t, \mathbf{y}) \qquad t \in (0, +\infty) \tag{2.7}$$

$$\mathbf{y}(0) = \xi_0 \tag{2.8}$$

has a solution $\mathbf{y}(t, \xi_0)$ for $t \in [0, +\infty)$.

The simplest choice of A-stable linearly implicit method to integrate numerically (2.7), (2.8) is:

$$(I - h\Phi_n)(\mathbf{y}_{n+1} - \mathbf{y}_n) = h\varphi_n \qquad n = 0, 1, 2, \ldots \tag{2.9}$$

$$\mathbf{y}_0 = \xi_0 \tag{2.10}$$

where $\mathbf{y}_n$ is the numerically computed approximation of $\mathbf{y}(nh, \xi_0)$, I is the identity matrix acting on $\mathbb{R}^m$, $h > 0$ is the stepsize, for $n = 0, 1, 2, \ldots$ $t_n = nh$, $\varphi_n = \varphi(t_n, \mathbf{y}_n)$, $\Phi_n = \Phi(t_n, \mathbf{y}_n)$ where $\Phi(t, \mathbf{y}) = \partial\varphi/\partial\mathbf{y}$ is the jacobian of φ with respect to $\mathbf{y}$. We note that when $\varphi(t, \mathbf{y}) = \Lambda\mathbf{y}$ is a linear map (2.9) reduces to the backward Euler method.

After rewriting (2.3) as a first-order system

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \tag{2.11}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{g}{\mu}D\mathbf{v} - \frac{1}{\mu}\nabla F(\mathbf{x}) \tag{2.12}$$

formulae (2.9), (2.10) with variable stepsize $h_n, n = 0, 1, \ldots$ (i.e. $t_0 = 0$, $t_n = \sum_{i=0}^{n-1} h_i$, $n = 1, 2, \ldots$) are applied to (2.11), (2.12), (2.4), (2.5). In this case the map $\varphi\colon \mathbb{R}^{2N} \longrightarrow \mathbb{R}^{2N}$ will be given by

$$\varphi\colon \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{v} \\ -\frac{g}{\mu}D\mathbf{v} - \frac{1}{\mu}\nabla F(\mathbf{x}) \end{bmatrix} \tag{2.13}$$

so that its jacobian matrix is given by

$$\Phi(\mathbf{x}) = \begin{bmatrix} 0 & I \\ -\frac{1}{\mu}L(\mathbf{x}) & -\frac{g}{\mu}D \end{bmatrix} \tag{2.14}$$

where

$$L(\mathbf{x}) = 2\left[J(\mathbf{x})^T J(\mathbf{x}) + \sum_{i=1}^{N} f_i(\mathbf{x}) H_i(\mathbf{x})\right] \tag{2.15}$$

$J(\mathbf{x}) = \partial\mathbf{f}(\mathbf{x})/\partial\mathbf{x}$ is the jacobian of $\mathbf{f}$ with respect to $\mathbf{x}$ and $H_i(\mathbf{x})$ is the hessian of $f_i(\mathbf{x})$.

Let $\mathbf{s}_n = \mathbf{x}_{n+1} - \mathbf{x}_n$, $n = 0, 1, 2, \ldots$; after some simple algebra (2.9) becomes:

$$\left[L_n + \frac{1}{h_n}\left(\frac{\mu}{h_n} I + gD\right)\right] \mathbf{s}_n = -\nabla F_n + \frac{\mu}{h_n}\mathbf{v}_n \tag{2.16}$$

$$\mathbf{v}_{n+1} = \frac{\mathbf{s}_n}{h_n} \qquad n = 0, 1, 2, \ldots \tag{2.17}$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{s}_n \tag{2.18}$$

where $L_n = L(\mathbf{x}_n)$, $\nabla F_n = \nabla F(\mathbf{x}_n)$. In order to avoid the computation of $H_i(\mathbf{x})$, $i = 1, 2, \ldots, N$, at each iteration and since we are looking for points $\mathbf{x}^*$ such that $\mathbf{f}(\mathbf{x}^*) = 0$ the term $\sum_{i=1}^{N} f_i(\mathbf{x})$ in (2.15) is dropped so that $L(\mathbf{x})$ is substituted by

$$\tilde{L}(\mathbf{x}) = 2J^T(\mathbf{x})J(\mathbf{x}). \tag{2.20}$$

Equation (2.16) will be replaced by

$$A_n \mathbf{s}_n = \mathbf{b}_n \tag{2.21}$$

where

$$A(\mathbf{x}, h) = \tilde{L}(\mathbf{x}) + \frac{1}{h}\left[\frac{\mu}{h} I + gD\right] \tag{2.22}$$

and

$$A_n = A(\mathbf{x}_n, h_n) \tag{2.23}$$

$$\mathbf{b}_n = -\nabla F_n + \frac{\mu}{h_n}\mathbf{v}_n \tag{2.24}$$

we note that the matrix A_n is symmetric and positive definite.

We have the following theorem:

THEOREM 2.1. *Let* $\mathbf{f} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *be twice continuously differentiable,* $F(\mathbf{x}) = \mathbf{f}(\mathbf{x})^T\mathbf{f}(\mathbf{x})$ *and* $L(\mathbf{x})$ *be given by* (2.15). *Let* $\mathbf{x}^* \in \mathbb{R}^N$ *be such that* $\mathbf{f}(\mathbf{x}^*) = 0$, $J(\mathbf{x}^*)$ *is nonsingular (i.e.* $\mathbf{x}^*$ *is a nondegenerate solution of the system* (2.1)*) and the following Lipschitz conditions holds:*

$$\|L(\mathbf{x}) - L(\mathbf{x}^*)\| \leq \gamma\|\mathbf{x} - \mathbf{x}^*\| \qquad \forall\, \mathbf{x} \in S = \left\{\mathbf{x} \,\middle|\, \|\mathbf{x} - \mathbf{x}^*\| < \delta\right\} \tag{2.25}$$

for some constants γ *and* δ *greater than zero. In the iteration* (2.21), (2.17), (2.18) *let* $\{h_n\}, n = 0, 1, 2, \ldots$, *be a sequence of positive numbers such that*

$$\lim_{n\to\infty} h_n = \infty \tag{2.26}$$

then there exists $\bar{h} > 0$ *such that for* $h_n > \bar{h}$, $n = 0, 1, \ldots$, $\mathbf{x}^*$ *is a point of attraction of* (2.21), (2.17), (2.18) *and the rate of convergence is*

(i) Q-superlinear if $h_n^{-1} \leq \gamma_1\|\nabla F(\mathbf{x}_n)\|$, $\gamma_1 > 0$, $n > n_0$, *for some* γ_1, $n_0 > 0$.

(ii) Q-quadratic if $h_n^{-1} \leq \gamma_2\|\nabla F(\mathbf{x}_n)\|^2$, $\gamma_2 > 0$, $n > n_0$, *for some* γ_2, $n_0 > 0$.

PROOF. Let us rewrite (2.21), (2.17), (2.18) as

$$\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n, h_n) + \frac{\mu}{h_n h_{n-1}} A_n^{-1}(\mathbf{x}_n - \mathbf{x}_{n-1}) \qquad n = 0, 1, 2, \ldots \tag{2.27}$$

where

$$\mathbf{G}(\mathbf{x}, h) = \mathbf{x} - A(\mathbf{x}, h)^{-1}\nabla F(\mathbf{x}) \tag{2.28}$$

with the initial conditions $\mathbf{x}_0 = \boldsymbol{\xi}_0, \mathbf{x}_{-1} = \boldsymbol{\xi}_0 - h_{-1}\boldsymbol{\eta}_0$, and $h_{-1} = h_0$, that is (2.21), (2.17), (2.18) can be interpreted as a two-step iteration. Since $\mathbf{x}^*$ is a nondegenerate solution of the system (2.1) $\mathbf{x}^*$ is an isolated minimizer of $F(\mathbf{x})$ and $\nabla F(\mathbf{x}^*) = 0$. Moreover for $h > 0$ the symmetric matrix $A(\mathbf{x}, h)$ is positive definite so that $A(\mathbf{x}, h)^{-1}$ exists that is $\mathbf{G}(\mathbf{x}, h)$ is well defined for $\mathbf{x} \in \mathbb{R}^N$ and $h > 0$ and $\mathbf{x}^*$ is fixed point of $\mathbf{G}(\mathbf{x}, h)$.

Let $\beta = \|L(x^*)^{-1}\|$ and let $\varepsilon \in (0,(2\beta)^{-1})$ then there exists $\delta > 0$ and $\bar{h} > 0$ such that:

$$\|L(x^*) - A(x,h)\| \leq \varepsilon \qquad \forall x \in S = \left\{x \,\middle|\, \|x - x^*\| < \delta\right\} \quad \forall h > \bar{h} \tag{2.29}$$

In fact

$$\|L(x^*) - A(x,h)\| \leq \|L(x^*) - \tilde{L}(x)\| + \|\tilde{L}(x) - A(x,h)\|$$

since $L(x^*) = \tilde{L}(x^*)$ there exists δ such that:

$$\|L(x^*) - \tilde{L}(x)\| \leq \frac{1}{2}\varepsilon \qquad \forall\, x \in S$$

and for a suitable $\bar{h} > 0$

$$\|L(x) - A(x,h)\| = \frac{1}{h}\|\frac{\mu}{h}I + gD\| \leq \frac{1}{2}\varepsilon \qquad \forall\, h > \bar{h} \tag{2.30}$$

From (2.29) and the perturbation lemma (lemma 2.3.2 p.45 of ORTEGA and RHEINBOLDT [19]) it follows that $A(x,h)^{-1}$ satisfies

$$\|A(x,h)^{-1}\| \leq \alpha = \frac{\beta}{1 - \beta\varepsilon} \qquad \forall\, x \in S\,,\ \forall\, h > \bar{h} \tag{2.31}$$

Moreover

$$\|G(x,h) - x^*\| \leq \omega(x,h)\|x - x^*\| \qquad \forall\, x \in S\,,\ \forall\, h > \bar{h} \tag{2.32}$$

where

$$\omega(x,h) = \alpha\left[\|A(x,h) - L(x)\| + \|L(x) - \tilde{L}(x^*)\| + \|q(x)\|\right] \tag{2.33}$$

and

$$q(x) = \begin{cases} 0 & x = x^* \\ \dfrac{\|\nabla F(x) - \nabla F(x^*) - L(x^*)(x - x^*)\|}{\|x - x^*\|} & x \neq x^* \end{cases}$$

In fact

$$\|G(x,h)-x^*\| = \|A(x,h)^{-1}\left[A(x,h)(x-x^*)-\nabla F(x)\right]\| \leq$$
$$\leq \alpha\Big\{\left[\|A(x,h)-L(x)\|+\|L(x)-\bar{L}(x^*)\|\right]\|x-x^*\|+$$
$$+\|\bar{L}(x^*)(x-x^*)+\nabla F(x^*)-\nabla F(x)\|\Big\}$$

Moreover from (2.25) and proposition 3.2.5 p. 70 of [19] we have

$$\|q(x)\| \leq \alpha_1\|x-x^*\| \qquad \forall\, x\in S \tag{2.34}$$

Hence from (2.30), (2.25), (2.33) and (2.34) for some constants $\alpha_2,\alpha_3>0$ we have

$$\omega(x,h) \leq \alpha_2\frac{1}{h}+\alpha_3\|x-x^*\| \qquad \forall\, x\in S,\quad \forall\, h>\bar{h} \tag{2.35}$$

From (2.27), (2.31), (2.32) for $x_n, x_{n-1}\in S$ and $h_n, h_{n-1}>\bar{h}$ we have

$$\begin{aligned}
&\|x_{n+1}-x^*\| \leq \\
&\leq \|G(x_n,h_n)-x^*\|+\frac{\mu}{h_nh_{n-1}}\|A_n^{-1}\left[(x_n-x^*)+(x^*-x_{n-1})\right]\| \leq \\
&\leq \left[\omega(x_n,h)+\frac{\mu\alpha}{h_nh_{n-1}}\right]\|x_n-x^*\|+\frac{\mu\alpha}{h_nh_{n-1}}\|x_{n-1}-x^*\| \leq \\
&\leq \left[\alpha_3\delta+\alpha_2\frac{1}{\bar{h}}+\frac{\mu\alpha}{\bar{h}^2}\right]\|x_n-x^*\|+\frac{\mu\alpha}{\bar{h}^2}\|x_{n-1}-x^*\|
\end{aligned} \tag{2.36}$$

Moreover from (2.36) eventually changing the values of δ and $\bar{h}$ we have

$$\begin{aligned}
\gamma_3 &= \alpha_3\delta+\frac{\alpha_2}{\bar{h}}+\frac{\mu\alpha}{\bar{h}^2}<\frac{1}{2}\\
\gamma_4 &= \frac{\mu\alpha}{\bar{h}^2}<\frac{1}{2}
\end{aligned} \tag{2.37}$$

so that

$$\|x_{n+1}-x^*\| < \gamma_3\|x_n-x^*\|+\gamma_4\|x_{n-1}-x^*\| \tag{2.38}$$

with $\alpha_4 = \gamma_3 + \gamma_4 < 1$ that is $\mathbf{x}_{n+1} \in S$. In particular we have shown that

$$\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}^* \tag{2.39}$$

that is $\mathbf{x}^*$ is a point of attraction of (2.27).

In particular for $n > n_0 > 0$, $\mathbf{x}_n \in S$, using (2.36) the required order-of-convergence estimates follows from:

$$\begin{aligned} \|\mathbf{x}_{n+1} - \mathbf{x}^*\| \le \left[\alpha_2 \frac{1}{h_n} + \alpha_3 \|\mathbf{x}_n - \mathbf{x}^*\|\right] \|\mathbf{x}_n - \mathbf{x}^*\| + \\ + \frac{\mu\alpha}{h_n h_{n-1}} \|\mathbf{x}_n - \mathbf{x}_{n-1}\| \qquad \text{for} \quad n > n_0 > 0 \end{aligned} \tag{2.40}$$

and the fact that

$$\|\nabla F(\mathbf{x}_n)\| \le (\|L(\mathbf{x}^*)\| + \varepsilon)\|\mathbf{x}_n - \mathbf{x}^*\| \tag{2.41}$$

where $\lim_{n\to\infty} \varepsilon_n = 0$.

Using the method given by (2.21), (2.17), (2.18) requires the solution of the linear system (2.21) at each step. Computing the exact solution with a direct method such as Gaussian elimination is very expensive when a large number of unknowns is involved and may not be worthwhile when $\mathbf{x}_k$ is far from $\mathbf{x}^*$. In this case it seems natural to solve the linear system (2.21) by an iterative procedure and to accept an approximate solution. In particular since the matrix A_n is symmetric and positive definite we may use conjugate gradients. When the method given by (2.21), (2.17), (2.18) is used to solve (2.21) with an iterative procedure, accepting an approximate solution, we will describe this procedure as an inexact method.

Let $\hat{\mathbf{s}}_n$ be the approximate step computed by the iterative procedure when solving (2.21) and

$$\mathbf{r}_n = A_n \hat{\mathbf{s}}_n - \mathbf{b}_n \tag{2.42}$$

be the residual. When $\mathbf{r}_n = 0$ the linear system is solved exactly. Let us assume that the approximate step computed $\hat{\mathbf{s}}_n$ satisfies the following condition:

$$\|\mathbf{r}_n\| < \hat{\beta}_n \|\mathbf{b}_n\| \qquad n = 0, 1, \ldots \tag{2.43}$$

for some forcing sequence $\{\hat{\beta}_n\}$, $n = 0, 1, \ldots$ We have the following theorem:

THEOREM 2.2. *Let* $\mathbf{f} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *be twice continuously differentiable,* $F(\mathbf{x}) = \mathbf{f}(\mathbf{x})^T\mathbf{f}(\mathbf{x})$ *and* $L(\mathbf{x})$ *be given by* (2.15). *Let* $\mathbf{x}^* \in \mathbb{R}^N$ *be such that* $\mathbf{f}(\mathbf{x}^*) = 0$, $J(\mathbf{x}^*)$ *is nonsingular and the following Lipschitz condition holds:*

$$\|L(\mathbf{x}) - L(\mathbf{x}^*)\| \leq \gamma\|\mathbf{x} - \mathbf{x}^*\| \quad \forall\, \mathbf{x} \in S = \left\{\mathbf{x} \middle| \|\mathbf{x} - \mathbf{x}^*\| < \delta\right\} \tag{2.44}$$

for some constants γ, δ *greater than zero. In the iteration* (2.21), (2.17), (2.18) *let* $\{h_n\}$, $n = 0, 1, 2, \ldots$, *be a sequence of positive numbers and let the linear system* (2.21) *be solved approximately in such a way that the residuals* $\mathbf{r}_n$ *given by* (2.42) *satisfy the condition* (2.43) *for some forcing sequence* $\{\hat{\beta}_n\}$, $n = 0, 1, \ldots$. *If* $0 < \hat{\beta}_n \leq \beta_{\max} < 1$, $n = 0, 1, \ldots$, *then there exists* $\bar{h} > 0$ *such that if* $h_n > \bar{h}$, $n = 0, 1, \ldots$, *then* $\mathbf{x}^*$ *is a point of attraction of the inexact method* (2.21), (2.17), (2.18).

PROOF. Since $J(\mathbf{x}^*)$ is nonsingular and $L(\mathbf{x}^*) = 2J(\mathbf{x}^*)^T J(x^*)$ we define the following norm:

$$\|\mathbf{x}\|_* = \|L(\mathbf{x}^*)\mathbf{x}\| \qquad \forall\, \mathbf{x} \in \mathbb{R}^N \tag{2.45}$$

we have

$$\frac{1}{\mu_1}\|\mathbf{x}\| \leq \|\mathbf{x}\|_* \leq \mu_1\|\mathbf{x}\| \qquad \forall\, \mathbf{x} \in \mathbb{R}^N \tag{2.46}$$

where

$$\mu_1 = \max\left\{\|L(\mathbf{x}^*)\|, \|L(\mathbf{x}^*)^{-1}\|\right\} \tag{2.47}$$

Moreover it is easy to see that under the stated hypotheses for any $\varepsilon > 0$ there exists $\delta > 0$ and $\bar{h} > 0$ such that:

$$\|A(\mathbf{x}, h) - L(\mathbf{x}^*)\| \leq \varepsilon \quad \forall\, \mathbf{x} \in S = \left\{\mathbf{x} \middle| \|\mathbf{x} - \mathbf{x}^*\| < \delta\right\},\ h > \bar{h} \tag{2.48}$$

$$\|A(\mathbf{x}, h)^{-1} - L(\mathbf{x}^*)^{-1}\| \leq \varepsilon \quad \forall\, \mathbf{x} \in S = \left\{\mathbf{x} \middle| \|\mathbf{x} - \mathbf{x}^*\| < \delta\right\},\ h > \bar{h} \tag{2.49}$$

$$\|\nabla F(\mathbf{x}) - \nabla F(\mathbf{x}^*) - L(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*)\| \leq \varepsilon\|\mathbf{x} - \mathbf{x}^*\|$$

$$\forall\, \mathbf{x} \in S = \left\{\mathbf{x} \,\middle|\, \|\mathbf{x} - \mathbf{x}\|^* < \delta\right\} \tag{2.50}$$

We have

(2.51)

$$L(\mathbf{x}^*)(\hat{\mathbf{x}}_{n+1} - \mathbf{x}^*) = \left[I + L(\mathbf{x}^*)(A_n^{-1} - L(\mathbf{x}^*)^{-1})\right] \cdot$$

$$\cdot\left\{\mathbf{r}_n + (A_n - L(\mathbf{x}^*))(\hat{\mathbf{x}}_n - \mathbf{x}^*) - \left[-\mathbf{b}_n - \nabla F(\mathbf{x}^*) - L(\mathbf{x}^*)(\hat{\mathbf{x}}_n - \mathbf{x}^*)\right]\right\}$$

and taking norms:

$$\|\hat{\mathbf{x}}_{n+1} - \mathbf{x}^*\|_* \leq \left[1 + \|L(\mathbf{x}^*)\|\,\|A_n^{-1} - L(\mathbf{x}^*)^{-1}\|\right] \cdot$$

$$\cdot\left[\|\mathbf{r}_n\| + \|A_n - L(\mathbf{x}^*)\|\,\|\hat{\mathbf{x}}_n - \mathbf{x}^*\| + \right. \tag{2.52}$$

$$\left. + \|-\mathbf{b}_n - \nabla F(\mathbf{x}^*) - L(\mathbf{x}^*)(\hat{\mathbf{x}}_n - \mathbf{x}^*)\|\right]$$

from (2.24) if $\hat{\mathbf{x}}_n \in S$ and $h_n > \bar{h}$ using (2.48), (2.49), 2.50) we have:

$$\|\hat{\mathbf{x}}_{n+1} - \mathbf{x}^*\|_* \leq \left[1 + \mu_1\varepsilon\right]\left[\hat{\beta}_n\|\nabla F(\hat{\mathbf{x}}_n)\| + 2\varepsilon\|\hat{\mathbf{x}}_n - \mathbf{x}^*\| + \right.$$

$$\left. + \frac{\mu}{h_n h_{n-1}}(1 + \hat{\beta}_n)(\|\hat{\mathbf{x}}_n - \mathbf{x}^*\| + \|\mathbf{x}^* - \hat{\mathbf{x}}_{n-1}\|)\right] \tag{2.53}$$

moreover from

$$\nabla F(\hat{\mathbf{x}}_n) = L(\mathbf{x}^*)(\hat{\mathbf{x}}_n - \mathbf{x}^*) + [\nabla F(\hat{\mathbf{x}}_n) - \nabla F(\mathbf{x}^*) - L(\mathbf{x}^*)(\hat{\mathbf{x}}_n - \mathbf{x}^*)] \tag{2.54}$$

we have

$$\|\nabla F(\hat{\mathbf{x}}_n)\| \leq \|\hat{\mathbf{x}}_n - \mathbf{x}^*\|_* + \varepsilon\|\hat{\mathbf{x}}_n - \mathbf{x}^*\|\,. \tag{2.55}$$

Finally from (2.47), (2.53), (2.55) we have:

$$\|\hat{\mathbf{x}}_{n+1} - \mathbf{x}^*\|_*[1 + \mu_1\varepsilon]\left[\beta_{\max}(1 + \varepsilon\mu_1) + \varepsilon\mu_1\left(2 + \frac{\mu}{\bar{h}^2}\right)\right] \cdot$$

$$\cdot\|\hat{\mathbf{x}}_n - \mathbf{x}^*\|_* + [1 + \mu_1\varepsilon]\frac{\mu_1\mu}{\bar{h}^2}(1 + \beta_{\max})\|\hat{\mathbf{x}}_{n-1} - \mathbf{x}^*\| = \tag{2.56}$$

$$= \alpha_5\|\hat{\mathbf{x}}_n - \mathbf{x}^*\|_* + \alpha_6\|\hat{\mathbf{x}}_{n-1} - \mathbf{x}^*\|$$

where

$$\alpha_5 = [1 + \mu_1\varepsilon]\left[\beta_{\max}(1 + \varepsilon\mu_1) + \varepsilon\mu_1\left(2 + \frac{\mu}{\bar{h}^2}\right)\right]$$

(2.57)

$$\alpha_6 = (1 + \mu_1\varepsilon)(1 + \beta_{\max})\left(\frac{\mu\mu_1}{\bar{h}^2}\right)$$

choosing the values of ε and $\bar{h}$ so that $\alpha_5 + \alpha_6 < 1$ from (2.56) we have that if $\hat{\mathbf{x}}_n, \hat{\mathbf{x}}_{n-1} \in S$ then $\mathbf{x}_{n+1} \in S$ and

$$\lim_{n\to\infty} \hat{\mathbf{x}}_n = \mathbf{x}^*$$

THEOREM 2.3. *Let* $\mathbf{f} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *be twice continuously differentiable,* $F(\mathbf{x}) = \mathbf{f}(\mathbf{x})^T\mathbf{f}(\mathbf{x})$ *and* $L(\mathbf{x})$ *be given by* (2.15). *Let* $\mathbf{x}^* \in \mathbb{R}^N$ *be such that* $\mathbf{f}(\mathbf{x}^*) = 0$, $J(\mathbf{x}^*)$ *is nonsingular and the following Lipschitz condition holds:*

$$\|L(\mathbf{x}) - L(\mathbf{x}^*)\| \le \gamma\|\mathbf{x} - \mathbf{x}^*\| \qquad \forall\, \mathbf{x}\ S = \{\mathbf{x} | \|\mathbf{x} - \mathbf{x}^*\| < \delta\} \tag{2.58}$$

In the iteration (2.21), (2.17), (2.18) *let* $\{h_n\}$, $n = 0, 1, \ldots$, *be a sequence of positive numbers and let the linear system* (2.21) *be solved approximately in such a way that residuals* $\mathbf{r}_n$ *given by* (2.42) *satisfy the condition* (2.43) *for some forcing sequence* $\{\hat{\beta}_n,\ n = 0, 1, \ldots\}$, *such that* $0 < \hat{\beta}_n < \hat{\beta}_{\max} < 1$, $n = 0, 1, \ldots$. *Then there exists* $\bar{h}$ *such that if* $h_n > \bar{h}$, $n = 0, 1, \ldots, \mathbf{x}^*$ *is a point of attraction of the inexact method* (2.21), (2.17), (2.18) *and the rate of convergence is:*

(i) Q-superlinear if $h_n^{-1} \le \gamma_1\|\nabla F(\hat{\mathbf{x}}_n)\|$, $\gamma_1 > 0$, $n > n_0$ *for some* $\gamma_1, n_0 > 0$ *and* $\lim_{n\to\infty} \hat{\beta}_n = 0$

(ii) Q-quadratic if $h_n^{-1} \le \gamma_2\|\nabla F(\hat{\mathbf{x}}_n)\|^2$, $\gamma_2 > 0$, $n > n_0$ *and* $\hat{\beta}_n \le \gamma_2\|\nabla F(\hat{\mathbf{x}}_n)\|$, $\gamma_2 > 0$, $n > n_0$, *for some* $\gamma_2, n_0 > 0$.

PROOF. From Theorem 2.2 we have that $\mathbf{x}^*$ is a point of attraction of the inexact method (2.21), (2.17), (2.18) so that we can assume that $\lim_{n\to\infty} \hat{\mathbf{x}}_n = \mathbf{x}^*$ and it remains to prove the rate-of-convergence results.

We have:

$$\hat{\mathbf{x}}_{n+1} - \mathbf{x}^* = A_n^{-1}\Big\{\mathbf{r}_n + \Big[A_n - L(\mathbf{x}^*)\Big](\hat{\mathbf{x}}_n - \mathbf{x}^*) + \\ - \Big[-\mathbf{b}_n - \nabla F(\mathbf{x}^*) - L(\mathbf{x}^*)(\hat{\mathbf{x}}_n - \mathbf{x}^*)\Big]\Big\} \tag{2.59}$$

and taking norms

$$\|\hat{\mathbf{x}}_{n+1} - \mathbf{x}^*\| \le \|A_n^{-1}\|\Big[\|\mathbf{r}_n\| + \|A_n - L(\mathbf{x}^*)\|\,\|\hat{\mathbf{x}}_n - \mathbf{x}^*\| + \\ + \|\nabla F(\hat{\mathbf{x}}_n) - \nabla F(\mathbf{x}^*) - L(\mathbf{x}^*)(\hat{\mathbf{x}}_n - \mathbf{x}^*)\| + \\ + \frac{\mu}{h_n h_{n-1}}\|\mathbf{x}_n - \mathbf{x}_{n-1}\|\Big] \tag{2.60}$$

Let $\varepsilon, \delta, \bar{h}$ be chosen in such a way that (2.29), (2.34), (2.35) hold then there exists n_0' such that for $n > n_0' + 1$, $\hat{\mathbf{x}}_n \in S = \{x \mid \|\mathbf{x} - \mathbf{x}^*\| < \delta\}$ we have:

$$\|\hat{\mathbf{x}}_{n+1} - \mathbf{x}^*\| \le \alpha\Big[\hat{\beta}_n\|\nabla F(\hat{\mathbf{x}}_n)\| + \\ + \Big(\alpha_2\frac{1}{h_n} + \alpha_3\|\hat{\mathbf{x}}_n - \mathbf{x}^*\|\Big)\|\hat{\mathbf{x}}_n - \mathbf{x}^*\| + \\ + \alpha_1\|\hat{\mathbf{x}}_n - \mathbf{x}^*\|^2 + \frac{\mu}{\bar{h}h_n}(1 + \hat{\beta}_n)\|\hat{\mathbf{x}}_n - \hat{\mathbf{x}}_{n-1}\|\Big] \tag{2.61}$$

and the desired rate-of-convergence results follow from (2.41).

3 – Complementarity problems and nonlinear systems

Let $\mathbf{f} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be given, the complementarity problem associated with $\mathbf{f}$ is

$$\mathbf{x} \ge 0 \tag{3.1}$$

$$\mathbf{f}(\mathbf{x}) \ge 0 \tag{3.2}$$

$$\langle \mathbf{x}, \mathbf{f}(\mathbf{x})\rangle = 0 \tag{3.3}$$

and let $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ be a strictly increasing function such that $\theta(0) = 0$. In [17] MANGASARIAN has shown that $\mathbf{x}^* \in \mathbb{R}^N$ is a solution of the complementarity problem (3.1), (3.2), (3.3) if and only if $\mathbf{x}^*$ is a solution of the system of nonlinear equations

$$\mathbf{g}(\mathbf{x}) = 0 \tag{3.4}$$

where $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_N(\mathbf{x}))^T$ and

$$g_i(\mathbf{x}) = \theta(|f_i(\mathbf{x}) - x_i|) - \theta(f_i(\mathbf{x})) - \theta(x_i) \qquad i = 1, 2, \ldots, N \tag{3.5}$$

for later purposes let us introduce

$$G(\mathbf{x}) = \mathbf{g}(\mathbf{x})^T \mathbf{g}(\mathbf{x}) \tag{3.6}$$

DEFINITION 3.1: Let $\mathbf{x}^* \in \mathbb{R}^N$ be a solution of the complementarity problem (3.1), (3.2), (3.3) we will say that $\mathbf{x}^*$ is nondegenerate if $\mathbf{x}^* + \mathbf{f}(\mathbf{x}^*) > 0$.

DEFINITION 3.2: Let $\mathbf{f}$ be continuously differentiable and $J(\mathbf{x}) = \partial \mathbf{f}/\partial \mathbf{x}$ be the jacobian of $\mathbf{f}$ with respect to $\mathbf{x}$, if for $\bar{n} = 1, 2, \ldots, N$ each principal minor $((\partial f_i/\partial x_j)), i, j = 1, 2, \ldots, \bar{n}$, is nonsingular we say that $J(\mathbf{x})$ has nonsingular principal minors.

In [17] MANGASARIAN has shown that if $\mathbf{x}^*$ is a nondegenerate solution of the complementarity problem (3.1), (3.2), (3.3) such that $J(\mathbf{x}^*)$ has nonsingular principal minors and $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly increasing differentiable function such that $d\theta/dt(0) + d\theta/dt(t) > 0, \forall\, t > 0$, then $\mathbf{x}^*$ is a solution of the nonlinear system (3.4) and $\partial \mathbf{g}/\partial \mathbf{x}(\mathbf{x}^*)$ the jacobian of $\mathbf{g}$ with respect to $\mathbf{x}$ is nonsingular.

For simplicity we choose $\theta(t) = t/2$ so that in a neighborhood of a nondegenerate solution of the complementarity problem (3.1), (3.2), (3.3) the function $\mathbf{g}(\mathbf{x})$ given by (3.5) has the same regularity properties of $\mathbf{f}(\mathbf{x})$. Given the local character of the convergence theorems of section 2 this is satisfactory. In section 4 the method for solving nonlinear system described in section 2 will be applied to (3.4) with $\theta(t) = t/2$ for some test complementarity problems.

4 – Numerical experience

The inexact method (2.21), (2.17), (2.18) has been implemented as follows:

i) since A_n is symmetric and positive definite the linear system (2.21) has been solved by the conjugate gradient method (C.G.) introduced by FLETCHER and REEVES [20]. This procedure solves an $N \times N$ linear system in at most N steps. Hovewer we stop the conjugate gradient procedure after a number of steps which is usually considerably lower than N. In fact let $\mathbf{s}_n^{(k)}$ be the approximate value for the solution $\mathbf{s}_n$ of the linear system (2.21) obtained as the result of step k of the conjugate gradient iteration; the iteration is stopped after step m if

$$\|A_n \mathbf{s}_n^{(m)} - \mathbf{b}_n\| \leq \hat{\beta}_n \|\mathbf{b}_n\|$$

ii) We have chosen:

$$\xi_0 = \eta_0 = 0$$

$$\mu = g = 1$$

$$D = I \quad \text{(the identity matrix)}$$

$$\mathbf{s}_n^{(0)} = 0 \qquad n = 0, 1, \ldots$$

and the following very simple variation laws for the time integration step-length h_n and the forcing sequence $\hat{\beta}_n$:

$$h_{n+1} = \min(10 h_n, h_{\max}) \qquad n = 0, 1, 2, \ldots$$

with $h_0 = 1$, $h_{\max} = 10^{35}$

$$\hat{\beta}_{n+1}^2 = \hat{\alpha}_n \hat{\beta}_n^2 \qquad n = 0, 1, 2, \ldots$$

where $\hat{\beta}_0$ is given and $\hat{\alpha}_n$ is automatically chosen by the program among the two values 0.1 and 0.5.

iii) The program stops in any case the conjugate-gradients iteration after N steps in order to avoid possible non termination due to the finite arithmetic of the computer.

Finally the method given by (2.21), (2.17), (2.18) (i.e. exact solution of the linear system (2.21)) is obtained simply setting $\hat{\beta}_0 = 0$.

The stopping rule adopted is $G(\hat{\mathbf{x}}_n) \leq 10^{-10}$ for the inexact method and $G(\mathbf{x}_n) \leq 10^{-10}$ for the "exact" method (i.e. $\hat{\beta}_0 = 0$). These methods have been coded in the Pascal programming language and the program has been run on a Hewlett-Packard 9816 computer.

We have tested the proposed algorithm on three complementarity problems of which two are linear and one is nonlinear.

The first problem considered arises as a one-dimensional free-boundary problem in the lubrication theory of an infinite journal bearing, i.e. a rotating cylinder separated from a bearing surface by a thin film of lubricating fluid [21]. The finite-difference approximation used by CRYER in [21] leads to

PROBLEM A (called Problem 3D by Cryer): Find $\mathbf{x}, \mathbf{w} \in \mathbb{R}^N$ such that

$$\mathbf{w} = \mathbf{q} + M\mathbf{x}, \qquad \mathbf{w} \geq 0, \quad \mathbf{x} \geq 0, \tag{4.1}$$

$$\langle \mathbf{w}, \mathbf{x} \rangle = 0 \tag{4.2}$$

where $M = ((M_{ij}))$, $i, j = 1, 2, \ldots, N$, is an $N \times N$ matrix with elements M_{ij} given by

$$\begin{aligned} M_{ij} &= -(H_{i+1/2})^3, && \text{if } j = i+1, \\ M_{ij} &= [(H_{i+1/2})^3 + (H_{i-1/2})^3], && \text{if } j = i, \\ M_{ij} &= -(H_{i-1/2})^3, && \text{if } j = i-1, \\ M_{ij} &= 0 && \text{otherwise} \end{aligned} \tag{4.3}$$

and $\mathbf{q} = (q_1, q_2, \ldots, q_N)^T$ is a vector with elements q_i given by

$$q_i = \frac{T}{N+1}[H_{i+1/2} - H_{i-1/2}], \quad i = 1, 2, \ldots N \tag{4.4}$$

where

$$H_{i\pm 1/2} + H\left[\left(i \pm \frac{1}{2}\right)\frac{T}{N+1}\right] \tag{4.5}$$

and the function $H(y)$ is given by

$$H(y) = \frac{1}{\sqrt{\pi}}(1 + \varepsilon \cos \pi y) > 0 \tag{4.6}$$

with

$$T = 2, \quad \varepsilon = 0.8 \tag{4.7}$$

We note that the matrix M given by (4.3) is symmetric and positive-definite.

The second problem arises as a two-dimensional free-boundary problem in the theory of the steady-state fluid flow through porous media. Some of these problems can be formulated as a variational inequality after an ingenious transformation proposed by BAIOCCHI and others (ref. [13]). The discretization used on the "model problem" ([3], p. 4) leads to

PROBLEM B: Find $\mathbf{x}, \mathbf{w} \in \mathbb{R}^N$ such that

$$\mathbf{w} = \mathbf{q} + M\mathbf{x}, \qquad \mathbf{w} \geq 0, \quad \mathbf{x} \geq 0, \tag{4.8}$$

$$\langle \mathbf{w}, \mathbf{x} \rangle = 0 \tag{4.9}$$

where M, an $N \times N$ real matrix, and $\mathbf{q} = (q_1, q_2, \ldots, q_N)^T \in \mathbb{R}^N$ are defined below.

Given n_x, n_y (positive integers) and X, Y (positive real numbers), let

$$N = n_x n_y,$$
$$Dx = X/n_x + 1,$$
$$Dy = Y/n_y + 1,$$
$$a = Dy/Dx,$$

let A be the $n_x \times n_x$ tridiagonal matrix having all the main diagonal elements equal to $2(a + 1/a)$, and the paradiagonal elements (i.e. immediately above or below the main diagonal) equal to $-a$, and let B be the $n_x \times n_x$ diagonal matrix with diagonal elements equal to $-1/a$. The matrix M is an $n \times n$ matrix with a block-tridiagonal structure ($n_y \times n_y$

blocks), having each main-diagonal block equal to the matrix A, and each paradiagonal block equal to the matrix B. We note that M is a positive-definite symmetric matrix. The vector $\mathbf{q}$ is defined as follows. Given W $(0 < W < Y)$, and using the Kronecker symbol δ_{ij}, let

$$g_L(y) = \frac{1}{2}(Y - y)^2,$$

$$g_R(y) = \frac{1}{2}(W - y)^2, \qquad \text{if} \quad y < W,$$

$$g_R(y) = 0, \qquad \text{if} \quad y \geq W,$$

$$g_D(x) = Y^2/2 - (Y^2 - W^2)(x/2X),$$

$$g_U(x) = 0,$$

$$r_{ij} = -DxDy + \delta_{i1} a g_L(jDy) + \delta_{in_x} a g_R(jDy) +$$
$$+ \delta_{ij}(1/a) g_D(iDx) + \delta_{n_y j}(1/a) g_U(iDx),$$
$$i = 1, 2, \ldots, n_x, \quad j = 1, 2, \ldots, n_y.$$

The elements $q_1, q_2, \ldots, q_n$ of $\mathbf{q}$ are given by

$$q_k = r_{ij}, \quad \text{with} \quad k = (j-1)n_x + i \tag{4.10}$$

Our last problem, which is defined below, can be interpreted as a finite-difference approximation of a nonlinear variational inequality.

PROBLEM C: Find $\mathbf{x}, \mathbf{w} \in \mathbb{R}^N$ such that

$$\mathbf{w} = M\mathbf{x} + \mathbf{p}(\mathbf{x}) + \mathbf{q}, \qquad \mathbf{w} \geq 0, \quad \mathbf{x} \geq 0 \tag{4.15}$$

$$\langle \mathbf{w}, \mathbf{x} \rangle = 0 \tag{4.16}$$

The problem dimension N, the quantities Dx, Dy and the matrix M are defined as in problem B, given n_x, n_y, X, Y. The nonlinear term $\mathbf{p}(\mathbf{x})$ is a vector in $\mathbb{R}^N$ with components $p_i = x_i^3$, $i = 1, \ldots, N$. The vector $\mathbf{q} = (q_1, q_2, \ldots, q_n)^T$ is defined by equation (4.10) where $r_{ij} = DxDy \sin(2\pi i Dx/X)$, $i = 1, 2, \ldots, n_x$, $j = 1, 2, \ldots, n_y$.

The numerical results obtained with the previously described methods on Problem A, B, C are shown in Table 1, 2, 3 respectively.

Table 1 - Results of Problem A

	$\eta_0 = 1$		$\eta_0 = 0$	
	n. of steps (2.21)	total n. of C.G. steps	n. of steps (2.21)	total n. of C.G. steps
30	10	79	7	210
40	12	121	8	320
50	16	238	8	400
60	14	240	8	480
70	15	318	9	630
80	15	369	9	720
90	19	650	9	810
100	18	556	10	1000

Table 2 - Results of Problem B
(with $X = 1.62$, $Y = 3.22$, $W = 0.84$)

n_x	n_y	N	$\eta_0 = 1$		$\eta_0 = 0$	
			n. of steps (2.21)	total n. of C.G. steps	n. of steps (2.21)	total n. of C.G. steps
6	9	54	13	170	6	324
8	12	96	15	250	8	768
10	15	150	17	483	10	1500
12	18	216	19	746	12	2592
14	21	294	19	867	14	4116
20	30	600	34	2405	21	126000

Table 3 - Results of Problem C
(with $X = 5$, $Y = 5$)

n_x	n_y	N	$\eta_0 = 1$		$\eta_0 = 0$	
			n. of steps (2.21)	total n. of C.G. steps	n. of steps (2.21)	total n. of C.G. steps
5	5	25	5	37	4	100
10	10	100	6	99	5	500
15	15	225	8	278	6	1350
20	20	400	10	407	6	2400
25	25	625	10	535	8	5000
30	30	900	10	893		

In tables 1, 2, 3 the adavantage of using "inexact linear algebra" with respect to complete solution of the linear system for problems A, B, C is shown, and the advantage is increasing with the number of unknowns.

REFERENCES

[1] R.W. COTTLE - G.B. DANTZIG: *Complementary pivot theory of mathematical programming*, Linear Algebra and Application 1 (1968), 103-125.

[2] F.J. GOULD - J.W. TOLLE: *A unified approach to complementarity in optimization*, Discrete Math. 7 (1974), 225-271.

[3] C. BAIOCCHI - V. COMINCIOLI - L. GUERRI - G. VOLPI: *Free boundary problems in the theory of fluid flow through porous media: a numerical approach*, Calcolo 10 (1973), 1-86.

[4] G.T. HERMAN: *Image reconstruction from projections: the fundamentals of computerized tomography*, Academic Press, New York, 1980.

[5] G.T. HERMAN - A. LEHT - P.H. LUTZ: *Relaxation methods for image reconstruction*, Communications of the A.C.M. 21 (1978), 152-158.

[6] C.E. LEMKE: *Bimatrix equilibrium points and mathematical programming*, Management Sci. 11 (1956), 681-689.

[7] E. ALLGOWER - K. GEORG: *Simplicial and continuation methods for approximating fixed points and solutions to systems of equations*, SIAM Review, 22 (1980), 28-85.

[8] D.A. BAYER - J.C. LAGARIAS: *The nonlinear geometry of linear programming I. Affine and projective scaling trajectories*, preprint AT&T Bell Laboratories 1986.

[9] S. INCERTI - V. PARISI - F. ZIRILLI: *A new method for solving nonlinear simultaneous equations*, SIAM J. Numerical Analysis 16 (1979), 770-789.

[10] F. ZIRILLI: *The solution of nonlinear systems of equations by second order systems of o.d.e. and linearly implicit A-stable techniques*, SIAM J. Numerical Analysis 19 (1982), 800-816.

[11] F. ALUFFI-PENTINI - V. PARISI - F. ZIRILLI: *A differential-equations algorithm for nonlinear equations*, ACM Transactions on Mathematical Software 10 (1984), 299-316.

[12] F. ALUFFI-PENTINI - V. PARISI - F. ZIRILLI: *Algorithm 617: DAFNE-A differential-equations algorithm for nonlinear equations*, ACM Transactions on Mathematical Software 10 (1984), 317-324.

[13] F. ALUFFI-PENTINI - V. PARISI - F. ZIRILLI: *Global optimization and stochastic differential equations*, Journal of Optimization Theory and Applications 47 (1985), 1-17.

[14] F. ALUFFI-PENTINI - V. PARISI - F. ZIRILLI: *A global optimization algorithm using stochastic differential equations*, ACM Transactions on Mathematical Software 14 (1988), 345-365.

[15] F. ALUFFI-PENTINI - V. PARISI - F. ZIRILLI: *Algorithm 667: SIGMA - A stochastic-integration global minimization algorithm*, ACM Transactions on Mathematical Software 14 (1988), 366-380.

[16] R. DEMBO - S. EISENSTAT - T. STEIHAUG: *Inexact Newton methods*, SIAM J. Numerical Analysis 19 (1982), 400-408.

[17] O.L. MANGASARIAN: *Equivalence of the complementarity problem to a system of nonlinear equations*, SIAM J. Applied Mathematics 31 (1976), 89-92.

[18] F. ALUFFI-PENTINI - V. PARISI - F. ZIRILLI: *An inexact continuous method in complementarity problems*, in: Control Applications of Nonlinear Programming and Optimization, G. di Pillo (ed.), Pergamon Press. Oxford (1986), 19-26.

[19] J.M. ORTEGA - W.C. RHEINBOLDT: *Iterative solution of nonlinear equations in several variables*, Academic Press, New York, 1970.

[20] R. FLETCHER - C. REEVES: *Function minimisation by conjugate gradients*, The Computer Journal 7 (1964), 149-154.

[21] C. CRYER: *The method of Cristopherson for solving free boundary problems for infinite journal bearings by means of finite differences*, Math. of Computation 25 (1971), 435-443.

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APPENDIX 2

A Quadratically Convergent Method for Linear Programming*

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ABSTRACT

A new method to solve linear programming problems is introduced. This method follows a path defined by a system of o.d.e., and for nondegenerate problems is quadratically convergent.

1. INTRODUCTION

Let R^n be the n-dimensional real Euclidean space, and $x = (x_1, \ldots, x_n)^T \in R^n$, where the superscript T means transpose. For x, $y \in R^n$ let $x^T y$ be the usual Euclidean inner product, and let $e = (1, \ldots, 1)^T$.

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A linear programming problem consists in minimizing a linear function over a region defined by linear equality and inequality constraints. We will say that a linear programming problem is in canonical form when it is written as follows:

$$\underset{x \in R^n}{\text{minimize}}\ \mathbf{c}^T\mathbf{x} \qquad (1.1)$$

subject to

$$A\mathbf{x} = \mathbf{0}, \qquad (1.2)$$

$$\mathbf{e}^T\mathbf{x} = 1, \qquad (1.3)$$

$$\mathbf{x} \geqslant \mathbf{0}, \qquad (1.4)$$

with side conditions $A\mathbf{e} = \mathbf{0}$, where

$$A = \left((a_{ij})\right)_{i=1,\ldots,m,\ j=1,\ldots,n}, \qquad n \geqslant 2, \quad m < n,$$

and $\mathbf{c} \in R^n$ are given, and the inequality (1.4) is understood componentwise, that is,

$$x_j \geqslant 0, \qquad j = 1, \ldots, n.$$

Moreover, we assume that the matrix A is of rank m.

We note that on these hypotheses $(1/n)\mathbf{e}$ is a feasible point, so that the feasible region is not empty.

The simplex method applies to linear programming problems in standard form, that is,

$$\underset{y \in R^s}{\text{minimize}}\ \mathbf{d}^T\mathbf{y} \qquad (1.5)$$

subject to

$$C\mathbf{y} \geqslant \mathbf{b}, \qquad (1.6)$$

$$\mathbf{y} \geqslant \mathbf{0}, \qquad (1.7)$$

where $\mathbf{d} \in R^s$, $\mathbf{b} \in R^r$, $C \in R^{r \times s}$, $r \leqslant s$, are given, and the inequalities (1.6), (1.7) are understood componentwise. In [1] it has been shown that a linear

programming problem in standard form with a finite solution can always be reduced to canonical form. Moreover, Karmarkar assumes that the objective function of the problem (1.1)–(1.4) is such that

$$z^* = \mathbf{c}^T \mathbf{x}^* = 0$$

for any feasible point $\mathbf{x}^*$ that is a solution of the linear programming problem (1.1)–(1.4). The problem (1.1)–(1.4) with this extra assumption is called a problem in canonical form with a normalized objective function. In our work the assumption of having a normalized objective function is not necessary; however, since this assumption simplifies some of the following algebraic manipulations, we will keep it.

Let Ω denote the subspace $\Omega = \{\mathbf{x} \in R^n \mid A\mathbf{x} = 0\}$, let Δ be the simplex $\Delta = \{\mathbf{x} \in R^n \mid \mathbf{x} \geqslant 0,\ \mathbf{e}^T\mathbf{x} = 1\}$, and finally let

$$\Lambda = \Omega \cap \Delta \tag{1.8}$$

be the polytope of the feasible points. Then the problem (1.1)–(1.4) can be rewritten as follows:

$$\underset{\mathbf{x} \in \Lambda}{\text{minimize}}\ \mathbf{c}^T\mathbf{x}. \tag{1.9}$$

In this paper we will introduce a new method to solve linear programming problems in canonical form with a normalized objective function. This class of problems is the one considered by N. Karmarkar in his celebrated paper [2].

In the late 1940s G. B. Dantzig [3] developed the simplex method to solve linear programming problems. In 1972 V. Klee and G. L. Minty [4] showed that the worst case complexity of the simplex method is combinatorial. Here the term "complexity" means the number of elementary operations necessary to solve a linear programming problem in the standard form (1.5)–(1.7). Since the simplex method finds the solution after a finite number of iterations, Klee and Minty [4] were able to give an example where the simplex method has complexity

$$p = O(rs\,2^s).$$

Note that in (1.5) $\mathbf{y} \in R^s$. Moreover, in 1981 S. Smale in [5] showed that the

"average" complexity of the simplex method is

$$p = O(rs^2),$$

where r, s are the dimensions of the matrix C in (1.6).

In spite of its worst case combinatorial complexity, the simplex method has been very successful in solving linear programming problems. The feature of the simplex method responsible for its worst case combinatorial complexity is that it moves on the boundary of the feasible region

$$Q = \{y \in R^s | Cy \geq b, y \geq 0\}.$$

In recent years a great deal of effort has been spent in the attempt to find a new algorithm for linear programming whose complexity in the worst case is polynomial. It is believed that these new methods will go through the *interior* of the feasible region Q.

In 1979 L. G. Khachijan [6] introduced the first method of this class, called the *ellipsoid method*. The worst case complexity of this method is

$$p = O(s^6).$$

Here, however, the meaning of the term "complexity" has been slightly changed. In fact the ellipsoid method does not step after a finite number of iterations, so that "complexity" means the number of elementary operations necessary to arrive in a predetermined neighborhood of the solution. Moreover, the method introduced by Khachijan is only of theoretical interest, since its practical performance is rather poor.

In 1984 N. Karmarkar [2] presented a new linear programming method of polynomial worst case complexity

$$p = O(s^{3.5}).$$

This algorithm is called the *projective method* when applied to a linear programming problem in canonical form with a normalized objective function. This algorithm is of theoretical and practical importance.

Since 1984 a great deal of work has been done in developing new methods for linear programming. Several "interior point algorithms" have been proposed. P. E. Gill, W. Murray, M. A. Saunders, J. A. Tomlin, and M. H. Wright in [7] have interpreted Karmarkar's algorithm as a "logarithmic

barrier method" and have suggested a new algorithm with good practical performance.

In the framework of logarithmic barrier function methods we can recall the work of several authors. In [8] J. Renegar lowered Karmarkar's complexity bound. In [9] C. Gonzaga lowered Renegar's complexity bound. In [10] M. Iri and H. Imai, with the hypothesis of being able to perform exact line-searches, introduced a quadratically convergent algorithm for the linear programming problem. In [11] N. Megiddo studied the geometrical properties of the paths derived from "weighted logarithmic barrier functions." Finally, J. A. Tomlin in [12] reports on considerable numerical experimentation with this kind of algorithms.

In this paper, as suggested by D. A. Bayer and J. C. Lagarias in [13], we will show that Karmarkar's projective method can be obtained by applying Euler's method with variable stepsize to a suitable initial value problem for a system of ordinary differential equations. In fact Karmarkar's method obtains the solution $\mathbf{x}^*$ of the linear programming problem by computing

$$\lim_{t \to \infty} \mathbf{x}\left(t, \frac{1}{n}\mathbf{e}\right). \tag{1.10}$$

where $\mathbf{x}(t, (1/n)\mathbf{e})$ is the solution of a system of ordinary differential equations with initial condition $(1/n)\mathbf{e}$, using Euler's method with variable stepsize. The idea of obtaining the solution of nonlinear programming problems as limit points of the trajectories of systems of ordinary differential equations has been widely used; for a review see [14]. In particular, in [15–17] quadratically convergent algorithms for nonlinear systems of equations have been obtained from methods based on the numerical integration of trajectories of systems of ordinary differential equations.

The interpretation of Karmarkar's projective method as the numerical solution of an initial value problem raises two natural questions:

(i) Can the system of ordinary differential equations used in Karmarkar's projective method be changed to a new one that will generate an interesting algorithm?

(ii) Can the Euler method with variable stepsize that is used in Karmarkar's projective method be replaced with some other numerical scheme that will generate interesting algorithms?

An answer to question (i) has been given by D. A. Bayer and J. C. Lagarias in [13] and J. L. Nazareth in [18], who replaced Karmarkar's vector field with the *affine vector field*. Question (ii) has been considered by N. Karmarkar, J. C. Lagarias, L. Slutsman, and P. Wang in [19], where they tried

to approximate the path $x(t,(1/n)e)$ with a power series expansion, obtaining encouraging practical results. In this paper we give two new answers to questions (i) and (ii); in fact, we propose a vector field which is different from the ones previously considered, and we use a linearly implicit A-stable integration scheme [14] to solve the initial value problem considered. In this way we obtain a quadratically convergent algorithm for linear programming problem. Moreover our algorithm shows good practical behavior.

In Section 2 Karmarkar's projective method is interpreted as the numerical integration of an initial value problem with Euler's method and variable stepsize. Moreover, to a linear programming problem in canonical form with normalized objective function is associated a new system of ordinary differential equations. If we assume that the solution of the linear programming problem is unique, this solution can be obtained as the limit point of a suitable trajectory of the system of ordinary differential equations.

In Section 3 an initial value problem for this system of ordinary differential equations is integrated numerically, using a linearly implicit A-stable method with variable stepsize. It is shown that this is a quadratically convergent algorithm for linear programming.

Finally, in Section 4 we compare the computational cost of our step with that of Karmarkar's projective algorithm and that of the simplex algorithm, and we present some numerical experiments.

2. THE USE OF ORDINARY DIFFERENTIAL EQUATIONS IN LINEAR PROGRAMMING

Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \in R^n$, $X^* \in R^{n \times n}$ be given by $X^* = ((X_{ij}^*)) = \mathrm{Diag}(x^*)$, that is, $X_{ij}^* = x_i^* \delta_{ij}$, $i, j = 1, 2, \ldots, n$, where δ_{ij} is the Kronecker symbol.

DEFINITION 2.1. A minimizer x^* of the problem (1.1)–(1.4) is called nondegenerate if it has exactly $n - m - 1$ null components.

Let $J_n = (1, 2, \ldots, n)$ and $S = (s_1, s_2, \ldots, s_{m+1})$, $m + 1 \leq n$, be an ordered set of indices such that $S \subseteq J_n$. Let $z = (z_1, z_2, \ldots, z_n) \in R^n$ be a vector. We denote by z_S the vector $z_S = (z_{s_1}, z_{s_2}, \ldots, z_{s_{m+1}}) \in R^{m+1}$. Moreover, given a vector $v \in R^{m+1}$ and a matrix $Q \in R^{(m+1) \times n}$ of rank $m + 1$, we denote by Q_S the submatrix $Q_S = [q^{s_1}, q^{s_2}, \ldots, q^{s_{m+1}}] \in R^{(m+1) \times (m+1)}$, where q^j is the jth column of Q. If B is an ordered set of indices such that $B \subseteq J_n$ and $N = J_n - B$, then the system

$$Qz = v \tag{2.1}$$

can be rewritten in the following form:

$$Q_B z_B + Q_N z_N = v. \tag{2.2}$$

DEFINITION 2.2. Let B be an ordered set of $m+1$ indices. Then B is a set of basic indices for the system (2.2) if there exists a matrix $\bar{Q}_N \in R^{(m+1)\times(n-m-1)}$ and a vector $\bar{v} \in R^{m+1}$ such that the system (2.2) is equivalent to the system

$$z_B + \bar{Q}_N z_N = \bar{v}. \tag{2.3}$$

LEMMA 2.3. *Let B be an ordered set of $m+1$ indices such that B is a set of basic indices for the system (2.2). Then Q_B is an invertible matrix.*

Proof. It follows immediately from the equivalence of the linear systems (2.2) and (2.3). ∎

Let $\alpha > 0$, $x \geqslant 0$, $x^\alpha = (x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha)^T \in R^n$, and $X^\alpha \in R^{n\times n}$ be the matrix $X^\alpha = \mathrm{Diag}(x^\alpha)$.

LEMMA 2.4. *Let $\alpha \geqslant 1$ and x^* be a nondegenerate minimizer of the linear programming problem (1.1)–(1.4). Then $AX^{*\alpha}A^T$ is an invertible matrix.*

Proof. Let

$$M = \begin{bmatrix} A \\ \hline e^T \end{bmatrix} \in R^{(m+1)\times n}$$

be the matrix A with the extra row e^T added. Since x^* is a nondegenerate minimizer of the problem (1.1)–(1.4) and M has rank $m+1$, then there exists an ordered set of $m+1$ indices B such that $x_B^* \in R^{m+1}$ has all nonzero components and $x_N^* \in R^{n-m-1}$ is the zero vector, where $N = J_n - B$. Moreover, B is a set of basic indices for the system

$$My = u, \tag{2.4}$$

where $u \in R^{m+1}$ is given and $y \in R^n$. Let $X_B^{*1/2} \in R^{(m+1)\times(m+1)}$ be the

matrix $X_B^{*1/2} = \mathrm{Diag}(x_B^{*1/2})$, and $X_N^{*1/2} \in R^{(n-m-1)\times(n-m-1)}$ be the matrix $X_N^{*1/2} = \mathrm{Diag}(x_N^{*1/2})$, that is, the null matrix. So from Lemma 2.3, $M_B \in R^{(m+1)\times(m+1)}$ is invertible, which implies that $M_B X_B^{*1/2}$ is invertible. Moreover, since B is a set of basic indices for the system (2.4), we have

$$MX^*M^T = \left(M_B X_B^{*1/2} + M_N X_N^{*1/2}\right)\left(X_B^{*1/2} M_B^T + X_N^{*1/2} M_N^T\right)$$

$$= \left(M_B X_B^{*1/2}\right)\left(X_B^{*1/2} M_B^T\right). \tag{2.5}$$

Since $M_B X_B^{*1/2}$ is invertible, from (2.5) it follows that MX^*M^T is invertible, so that an easy computation shows that AX^*A^T is invertible. Therefore it follows that $AX^{*1/2}$ is of rank m, so that $AX^{*\alpha/2}$ is of rank m and $AX^{*\alpha}A^T$ is invertible. ■

LEMMA 2.5. *Let $\alpha = 1$ or $\alpha = 2$, and let x^* be a nondegenerate minimizer of the linear programming problem* (1.1)–(1.4). *Then there exists $\rho^* > 0$ such that $AX^\alpha A^T$ is invertible for $x \in S(x^*, \rho^*)$, where $S(x^*, \rho^*) = \{x \in R^n \mid \|x - x^*\| < \rho^*\}$.*

Proof. The proof follows immediately from the continuity of $AX^\alpha A^T$ with respect to $x \in R^n$, from Lemma 2.4, and from J. M. Ortega and W. C. Rheinboldt [20, Lemma 2.3.2, p. 45]. ■

Let $x = (x_1, x_2, \ldots, x_n)^T \in R^n$; let $X \in R^{n\times n}$ be given by $X = \mathrm{Diag}(x)$; let $D \in R^{m\times n}$, $m \leqslant n$, be a matrix, and D^- the subspace

$$D^- = \{x \in R^n \mid Dx = 0\}. \tag{2.6}$$

and let $\Pi_{D^-}(\ )$ be the orthogonal projection on D^-. The projector $\Pi_{D^-}(\ )$ always exists, and if D has full rank is given by

$$\Pi_{D^-}(y) = \left[I - D^T(DD^T)^{-1}D\right]y, \qquad y \in R^n. \tag{2.7}$$

Let Γ be

$$\Gamma = \{x \in R^n \mid x \geqslant 0\}. \tag{2.8}$$

and $\mathring{\Gamma}$ be its interior.

$$\mathring{\Gamma} = \{x \in R^n | x > 0\}. \tag{2.9}$$

The set $\mathring{\Gamma}$ is called the positive orthant.

For $\alpha > 0$ and $x \in \Gamma$ let $X^{\alpha/2} \in R^{n \times n}$ be the matrix $X^{\alpha/2} = \text{Diag}(x_1^{\alpha/2}, x_2^{\alpha/2}, \ldots, x_n^{\alpha/2})$. We observe that $\Pi_{(AX^{\alpha/2})}$ can always be expressed in the form (2.7). In fact, let r be the rank of the matrix $AX^{\alpha/2}$: if $r = m$, then $\Pi_{(AX^{\alpha/2})}$ is given by (2.7). If $0 < r < m$, we can consider the matrix $\bar{A} \in R^{r \times n}$ obtained from A by eliminating the $m - r$ rows of A with indices equal to those of the $m - r$ rows of $AX^{\alpha/2}$ that are linearly dependent. Since $(AX^{\alpha/2})^T = (\bar{A}X^{\alpha/2})^T$, we have

$$\Pi_{(AX^{\alpha/2})} = \Pi_{(\bar{A}X^{\alpha/2})}. \tag{2.10}$$

where $\Pi_{(\bar{A}X^{\alpha/2})}$ is given by (2.7). Finally if $r = 0$ we have that $\Pi_{(AX^{\alpha/2})} = I$, where I is the $n \times n$ identity matrix. Let $h(x) \in R^n$ be the following vector field:

$$h(x) = -X(I - ee^T X)\Pi_{(AX)}(Xc), \qquad x \in R^n. \tag{2.11}$$

The vector field $h(x)$ is known as Karmarkar's vector field [2, 13]. Let $S(x^*, \rho^*)$ be the open ball of Lemma 2.5, and let us consider $h(x)$ for $x \in \Gamma \cup S(x^*, \rho^*)$.

We observe that for $x \in \mathring{\Gamma} \cup S(x^*, \rho^*)$, AX is of rank m. Then $h(x)$ is a continuously differentiable function of x for $x \in \mathring{\Gamma} \cup S(x^*, \rho^*)$. Let Λ be given by (1.8), and $\mathring{\Lambda}$ be given by

$$\mathring{\Lambda} = \Lambda \cap \mathring{\Gamma}. \tag{2.12}$$

We will consider the initial value problem

$$\frac{dx}{dt} = h(x), \tag{2.13}$$

$$x(0) = \frac{1}{n}e. \tag{2.14}$$

It is easy to verify that $(1/n)e \in \mathring{\Lambda}$.

LEMMA 2.6. *Let*

$$B = \begin{bmatrix} AX \\ \mathbf{e}^T \end{bmatrix} \in R^{(m+1)\times n}$$

be the matrix AX with the extra row $\mathbf{e}^T$ added, and let $\gamma = \alpha/\sqrt{n(n-1)}$, where $\alpha \in (0,1)$ is a parameter. For $\mathbf{x}^k \in \mathring{\Lambda}$ let

$$X_k = \mathrm{Diag}(\mathbf{x}^k), \qquad B_k = \begin{bmatrix} AX_k \\ \mathbf{e}^T \end{bmatrix},$$

and let Δt_k be given by

$$\Delta t_k = \left(\frac{1}{n\gamma} \| \Pi_{B_k^T}(X_k \mathbf{c}) \| - \mathbf{e}^T X_k \Pi_{B_k^T}(X_k \mathbf{c}) \right)^{-1}. \tag{2.15}$$

Then Euler's method applied to the initial value problem (2.13), (2.14) with variable stepsize Δt_k given by (2.15) produces the sequence $\{\mathbf{x}^k\}$, $k = 0, 1, 2, \ldots$, generated by Karmarkar's algorithm [2, pp. 378–379] applied to the linear programming problem with normalized objective function (1.1)–(1.4).

Proof. We observe that $\Delta t_k > 0$ for $\mathbf{x}^k \in \mathring{\Lambda}$ (see [2, Theorem 5, pp. 381–382], so that, integrating (2.13), (2.14) with Euler's method and variable stepsize Δt_k, we have

$$\mathbf{x}^0 = \frac{1}{n}\mathbf{e} \tag{2.16}$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \Delta t_k \mathbf{h}(\mathbf{x}^k) \qquad k = 0, 1, 2, \ldots. \tag{2.17}$$

The thesis follows from a straightforward computation. ∎

For $\alpha > 0$ let

$$\mathbf{g}(\mathbf{x}, \alpha) = \Pi_{(AX^\alpha)^T}(X^\alpha \mathbf{c}), \qquad \mathbf{x} \in \Gamma. \tag{2.18}$$

We note that for $\alpha = 2$, $\mathbf{g}(\mathbf{x}, 2)$ is the affine scaling factor of [13], and

$$\mathbf{g}(\mathbf{x}, 2) = \Pi_{(AX)^T}(X\mathbf{c}) \tag{2.19}$$

can be defined for $\mathbf{x} \in R^n$ so that

$$\mathbf{h}(\mathbf{x}) = -X(I - \mathbf{e}\mathbf{e}^T X)\mathbf{g}(\mathbf{x}, 2), \qquad \mathbf{x} \in R^n. \tag{2.20}$$

We have:

THEOREM 2.7. *For $0 < \alpha \leqslant 2$ let $\mathbf{x} \in \Lambda$ be a feasible point for the linear programming problem with normalized objective function* (1.1)–(1.4). *Then*

$$\Pi_{(AX^\alpha)^T}(X^\alpha \mathbf{c}) = \mathbf{0} \tag{2.21}$$

if and only if $\mathbf{x}$ is a minimizer of the linear programming problem with normalized objective function (1.1)–(1.4).

Proof Let $\mathbf{x} = X\mathbf{e}$ be a minimizer of the problem (1.1)–(1.4) with normalized objective function. Then

$$\Pi_{(AX^\alpha)^T}(X^{\alpha/2}\mathbf{c}) = \mathbf{0}. \tag{2.22}$$

In fact, if we assume that

$$\Pi_{(AX^\alpha)^T}(X^{\alpha/2}\mathbf{c}) \neq \mathbf{0}, \tag{2.23}$$

then there exists $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in R^n$ such that

$$\sum_{j=1}^{n} a_{ij} x_j^{\alpha/2} z_j = 0, \qquad i = 1, \ldots, m, \tag{2.24}$$

that is, $\mathbf{z} \in (AX^{\alpha/2})^-$ and

$$\sum_{j=1}^{n} c_j x_j^{\alpha/2} z_j = \beta, \qquad \beta \neq 0. \tag{2.25}$$

that is, $X^{\alpha/2}\mathbf{c}$ is not orthogonal to $\mathbf{z}$. We can assume without loss of generality $\beta > 0$. Let us define

$$w_j = x_j^{\alpha/2} z_j, \qquad j = 1, 2, \ldots, n. \tag{2.26}$$

Since $\beta > 0$, there exists j such that $w_j \neq 0$. If $w_j \leqslant 0$ for $j = 1, 2, \ldots, n$, we choose $\varepsilon > 0$; otherwise we choose ε as follows:

$$0 < \varepsilon < \min_{j:\, w_j > 0} \frac{x_j}{w_j}. \tag{2.27}$$

We recall that $x_j \geqslant 0$ for $j = 1, 2, \ldots, n$. Let

$$v_j = x_j - \varepsilon w_j, \qquad j = 1, 2, \ldots, n. \tag{2.28}$$

From (2.27) we have

$$v_j \geqslant 0, \qquad j = 1, 2, \ldots, n. \tag{2.29}$$

and

$$\sum_{j=1}^{n} v_j > 0. \tag{2.30}$$

Let us define

$$u_j = \frac{v_j}{\sum_{i=1}^{n} v_i}, \qquad j = 1, 2, \ldots, n. \tag{2.31}$$

The point $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$ is a feasible point; in fact,

$$A\mathbf{u} = \mathbf{0}, \tag{2.32}$$

$$\mathbf{e}^T\mathbf{u} = 1, \tag{2.33}$$

$$\mathbf{u} \geqslant \mathbf{0}. \tag{2.34}$$

Moreover,

$$\sum_{j=1}^{n} c_j u_j = \frac{1}{e^T v} \sum_{j=1}^{n} (c_j x_j - \varepsilon c_j w_j)$$

$$= \frac{1}{e^T v} \left(\sum_{j=1}^{n} c_j x_j - \varepsilon \sum_{j=1}^{n} c_j x_j^{\alpha/2} z_j \right)$$

$$= \frac{1}{e^T v} \left(\sum_{j=1}^{n} c_j x_j - \varepsilon \beta \right) = \frac{1}{e^T v} (c^T x - \varepsilon \beta). \tag{2.35}$$

Since x has been assumed to be a minimizer of the linear programming problem (1.1)–(1.4) and the objective function is normalized, we have

$$c^T x = 0. \tag{2.36}$$

Therefore the objective function assumes a negative value at u, and this is absurd.

Let us assume now that $x \in A$ and that Equation (2.21) holds. We will show that $x = Xe$ is a minimizer of the linear programming problem with normalized objective function (1.1)–(1.4). In fact, from (2.21) we have

$$e^T X^{1-\alpha/2} \Pi_{(AX^{\alpha/2})^\perp} (X^{\alpha/2} c) = 0. \tag{2.37}$$

Using $\bar{A}$ instead of A as in (2.10), when $AX^{\alpha/2}$ is of rank less than m we have

$$0 = e^T X^{1-\alpha/2} \Pi_{(AX^{\alpha/2})^\perp} (X^{\alpha/2} c)$$

$$= e^T X^{1-\alpha/2} \left[I - X^{\alpha/2} A^T (AX^\alpha A^T)^{-1} AX^{\alpha/2} \right] X^{\alpha/2} c$$

$$= e^T Xc - e^T XA^T (AX^\alpha A^T)^{-1} AX^\alpha c = e^T Xc = c^T x. \tag{2.38}$$

Therefore we have that x is a feasible point where $c^T x = 0$, that is, x is a minimizer for the linear programming problem with normalized objective function (1.1)–(1.4). ■

Let Σ be the set given by

$$\Sigma = \{x \in R^n | Ax = 0, e^T x = 1\}. \tag{2.39}$$

LEMMA 2.8. *Let x^* be a nondegenerate minimizer of the linear programming problem with normalized objective function* (1.1)–(1.4), *and let* $h(x)$ *be given by* (2.11). *Then we have*

$$h(x^*) = 0. \tag{2.40}$$

Moreover,

$$Ah(x) = 0, \qquad x \in \Sigma. \tag{2.41}$$

$$e^T h(x) = 0, \qquad x \in \Sigma. \tag{2.42}$$

where Σ is given by (2.39).

Proof. In fact for $x \in \Sigma$ we have $AXe = 0$, $e^T Xe = 1$, and $AX\Pi_{(AX)^-}(Xc) = 0$, so that

$$Ah(x) = -AX\Pi_{(AX)^-}(Xc) + AXee^T X\Pi_{(AX)^-}(Xc) = 0 \tag{2.43}$$

and

$$e^T h(x) = -e^T X\Pi_{(AX)^-}(Xc) + e^T Xee^T X\Pi_{(AX)^-}(Xc) = 0. \tag{2.44}$$

Moreover, from Theorem 2.7 we have (2.40). ∎

Let $x \in R^n$, and $E_1 \in R^{n \times n}$ be the matrix given by

$$E_1 = \mathrm{Diag}(\Pi_{(AX)^-}(Xc)), \qquad x \in R^n. \tag{2.45}$$

Let $J_h(x) \in R^{n \times n}$ be the following matrix:

$$J_h(x) = -(I - Xee^T)X\Pi_{(AX)^-}X^{-1}E_1 - \left[e^T X\Pi_{(AX)^-}(Xc)\right]I, \qquad x \in R^n. \tag{2.46}$$

For $x \in R^n$ in (2.46) we will use $\bar{A}$ instead of A, as in (2.10), when AX is of rank less than m. An elementary computation shows that the matrix $X\Pi_{(AX)^\perp}X^{-1}$ can be defined for $x \in R^n$ so that $J_h(x)$ is defined for $x \in R^n$.

Let

$$\mathcal{H} = \{x \in R^n \mid AX^2A^T \text{ is invertible}\}. \tag{2.47}$$

For $x \in \mathcal{H}$, $J_h(x)$ is the Jacobian matrix of $h(x)$ with respect to x. Moreover let $S(x^*, \rho^*)$ be the open ball of Lemma 2.5; we observe that for $x \in \mathring{\Gamma} \cup S(x^*, \rho^*)$, since AX is of rank m, the matrix $X\Pi_{(AX)^\perp}X^{-1}$ is well defined and continuous. So $J_h(x)$ is continuous for $x \in \mathring{\Gamma} \cup S(x^*, \rho^*)$, and since $\Pi_{(AX^*)^\perp}(X^*c) \equiv 0$, we have

$$J_h(x^*) \equiv 0. \tag{2.48}$$

From Lemma 2.5, we conclude that any solution x^* of the linear programming problem with normalized objective function (1.1)–(1.4) is an equilibrium point of (2.13), that is, $h(x^*) = 0$.

However, due to the singular Jacobian of $h(x)$ at x^* [that is, to (2.48)], the use of a linearly implicit A-stable method to integrate the initial value problem (2.13), (2.14), as suggested in [8] in the context of nonlinear programming, will not produce a quadratically convergent method for linear programming. To overcome this difficulty we introduce a new vector field $f(x) \in R^n$ defined for $x \in R^n$ given by

$$f(x) = -(I - Xee^T)\left[Xc - XA^T(AXA^T)^{-1}AXc\right], \qquad x \in R^n, \tag{2.49}$$

where we use $\bar{A}$ instead of A, as in (2.10), if AXA^T A is of rank less than m.

Let us consider $f(x)$ for $x \in \mathring{\Gamma} \cup S(x^*, \rho^*)$. We observe that AXA^T is invertible. From (2.49) we have that $f(x)$ is a continuously differentiable function of x for $x \in \mathring{\Gamma} \cup S(x^*, \rho^*)$. For later purposes we observe that $f(x)$ for $x \in \Gamma$ can be rewritten as follows:

$$f(x) = -(I - Xee^T)X^{1/2}\Pi_{(AX^{1/2})^\perp}(X^{1/2}c), \qquad x \in \Gamma, \tag{2.50}$$

or

$$f(x) = -X^{1/2}(I - X^{1/2}ee^TX^{1/2})g(x, 1), \qquad x \in \Gamma. \tag{2.51}$$

From Equations (2.20), (2.51) and Theorem 2.7 it follows that if x^* is a

minimizer of the linear programming problem, then $\mathbf{f}(\mathbf{x}^*) = \mathbf{h}(\mathbf{x}^*) = \mathbf{0}$, that is, $\mathbf{x}^*$ is an equilibrium point of the vector fields $\mathbf{h}(\mathbf{x}), \mathbf{f}(\mathbf{x})$. For $\mathbf{x} \in \mathring{\Delta}$ the vector field $\mathbf{f}(\mathbf{x})$ can be obtained as the steepest descent vector associated to the function $\mathbf{c}^T\mathbf{x}$ with respect to a particular metric. In [13] D. A. Bayer and J. C. Lagarias have introduced the idea of looking at Karmarkar's vector field $\mathbf{h}(\mathbf{x})$ in terms of steepest descent directions.

Let $\mathbf{x}_0$ be a feasible point of the linear programming problem (1.1)–(1.4), and F_0 be the affine subspace

$$F_0 = \mathbf{x}_0 + \{\mathbf{v} \in R^n | A\mathbf{v} = \mathbf{0}, \mathbf{e}^T\mathbf{v} = 0\}. \tag{2.52}$$

LEMMA 2.9. *The vector field* $\mathbf{f}(\mathbf{x})$ *given by* (2.51) *is the steepest descent vector associated to the objective function* $b(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ *of the linear programming problem* (1.1)–(1.4) *restricted to* $F_0 \cap \mathring{\Gamma}$ *with respect to the Riemannian metric* $G(\mathbf{x}) = X^{-1} = \mathrm{Diag}(\mathbf{x}^{-1})$, *defined on the positive orthant* $\mathring{\Gamma}$, *where* F_0 *is given by* (2.52).

Proof. We consider the following transformation for $\mathbf{x} \in F_0 \cap \mathring{\Gamma}$:

$$\mathbf{x} = G^{-1/2}(\mathbf{x})\mathbf{y} = X^{1/2}\mathbf{y}. \tag{2.53}$$

We have

$$b(\mathbf{x}(\mathbf{y})) = (X^{1/2}\mathbf{c})^T\mathbf{y}, \tag{2.54}$$

and F_0 assumes the following form:

$$F_0 = \mathbf{x}_0 + \{\mathbf{u} \in R^n | AX^{1/2}\mathbf{u} = \mathbf{0}, \mathbf{e}^TX^{1/2}\mathbf{u} = 0\}. \tag{2.55}$$

The gradient vector of $b(\mathbf{x}(\mathbf{y}))$ with respect to $\mathbf{y}$ is

$$\frac{\partial b}{\partial \mathbf{y}} = X^{1/2}\mathbf{c}. \tag{2.56}$$

The gradient vector $\partial b/\partial \mathbf{y}$ projected on $\begin{bmatrix} AX^{1/2} \\ \mathbf{e}^TX^{1/2} \end{bmatrix}^{\wedge}$ is given by

$$\boldsymbol{\xi} = \Pi_{\begin{bmatrix} AX^{1/2} \\ \mathbf{e}^TX^{1/2} \end{bmatrix}} \cdot (X^{1/2}\mathbf{c}), \tag{2.57}$$

where we use $\bar{A}$ instead of A, as in (2.10), if AXA^T is of rank less than m. Since $AXe = 0$ from (2.57), using (2.7) we have

$$\xi = X^{1/2}c - X^{1/2}A^T(AXA^T)^{-1}AXc - X^{1/2}ee^TXc. \tag{2.58}$$

Since $e^TXA^T(AXA^T)^{-1}AXc = 0$, we have

$$\xi = (I - X^{1/2}ee^TX^{1/2})\Pi_{(AX^{1/2})^-}(X^{1/2}c). \tag{2.59}$$

Finally, applying (2.53) to ξ, we have that the gradient vector $\zeta(x)$ is given by

$$\zeta(x) = X^{1/2}\xi = X^{1/2}(I - X^{1/2}ee^TX^{1/2})\Pi_{(AX^{1/2})^-}(X^{1/2}c). \tag{2.60}$$

This concludes the proof. ■

Let Λ be given by (1.8).

LEMMA 2.10. *Let $x \in \Lambda$, and r be the rank of the matrix AXA^T. Then*

$$\Pi_{\left[\frac{AX^{1/2}}{e^TX^{1/2}}\right]^-}(X^{1/2}c) = (I - X^{1/2}ee^TX^{1/2})\Pi_{(AX^{1/2})^-}(X^{1/2}c), \tag{2.61}$$

where we use $\bar{A}$ instead of A, as in (2.10), when r is less than m.

Proof. Let $0 < r \leq m$. The projector $\Pi_{\left[\frac{AX^{1/2}}{e^TX^{1/2}}\right]^-}$ is defined as follows:

$$\Pi_{\left[\frac{AX^{1/2}}{e^TX^{1/2}}\right]^-} = I - \left[\frac{AX^{1/2}}{e^TX^{1/2}}\right]^T\left(\left[\frac{AX^{1/2}}{e^TX^{1/2}}\right]\left[\frac{AX^{1/2}}{e^TX^{1/2}}\right]^T\right)^{-1}\left[\frac{AX^{1/2}}{e^TX^{1/2}}\right]. \tag{2.62}$$

Let us compute the matrix

$$M = \left(\left[\frac{AX^{1/2}}{e^TX^{1/2}}\right]\left[\frac{AX^{1/2}}{e^TX^{1/2}}\right]^T\right)^{-1}.$$

Since $AXe = 0$ and $e^T Xe = 1$, we have

$$M = \left[\begin{array}{c|c} (AXA^T)^{-1} & 0 \\ \hline 0^T & 1 \end{array}\right]. \tag{2.63}$$

An elementary computation gives us

$$\Pi_{\left[\frac{AX^{1/2}}{e^T X^{1/2}}\right]^-}(X^{1/2}c) = X^{1/2}c - X^{1/2}A^T(AXA^T)^{-1}AXc - X^{1/2}ee^T Xc. \tag{2.64}$$

From $e^T XA^T = 0$ we have

$$q = X^{1/2}ee^T XA^T(AXA^T)^{-1}AXc \equiv 0 \tag{2.65}$$

and

$$\Pi_{\left[\frac{AX^{1/2}}{e^T X^{1/2}}\right]^-}(X^{1/2}c) = \Pi_{(AX^{1/2})^-}(X^{1/2}c) - X^{1/2}ee^T X^{1/2}c + q. \tag{2.66}$$

With an easy computation from (2.66) we obtain (2.61). Let $r = 0$, and $O \in R^{n\times n}$ be the null matrix. We have that

$$\left[\frac{O}{e^T X^{1/2}}\right]^- = (e^T X^{1/2})^- \quad \text{and} \quad \Pi_{O^-} = I,$$

so that (2.61) holds. ■

LEMMA 2.11. *Let x^* be a nondegenerate minimizer of the linear programming problem with normalized objective function (1.1)–(1.4), let $f(x)$ be given by (2.49) and Σ be given by (2.39). Then we have*

$$f(x^*) = 0. \tag{2.67}$$

moreover

$$Af(x) = 0, \qquad x \in \Sigma. \tag{2.68}$$

$$e^T f(x) = 0, \qquad x \in \Sigma. \tag{2.69}$$

Proof. Let $\alpha = 1$. From Theorem 2.7 we have

$$\Pi_{(AX^{-1/2})^T}(X^{-1/2}\mathbf{c}) = \mathbf{0}, \tag{2.70}$$

so that

$$\mathbf{f}(\mathbf{x}^*) = \mathbf{0}. \tag{2.71}$$

Let $\mathbf{x} \in \Sigma$. Since $AX\mathbf{e} = \mathbf{0}$ and $\mathbf{e}^T X\mathbf{e} = 1$, we have

$$A\mathbf{f}(\mathbf{x}) = -(A - AX\mathbf{e}\mathbf{e}^T)\left[X\mathbf{c} - XA^T(AXA^T)^{-1}AX\mathbf{c}\right] = \mathbf{0} \tag{2.72}$$

and

$$\mathbf{e}^T\mathbf{f}(\mathbf{x}) = -(\mathbf{e}^T - \mathbf{e}^T X\mathbf{e}\mathbf{e}^T)\left[X\mathbf{c} - XA^T(AXA^T)^{-1}AX\mathbf{c}\right] = 0. \tag{2.73}$$

This concludes the proof. ∎

LEMMA 2.12. *Let* $\mathbf{f}(\mathbf{x})$ *be given by* (2.49), *and* $\mathbf{x}_0 \in R^n$ *be such that*

$$A\mathbf{x}_0 = \mathbf{0}, \tag{2.74}$$

$$\mathbf{e}^T\mathbf{x}_0 = 1. \tag{2.75}$$

Then the solution $\mathbf{x}(t)$ *of the initial value problem*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \tag{2.76}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{2.77}$$

satisfies the constraints

$$A\mathbf{x}(t) = \mathbf{0}, \tag{2.78}$$

$$\mathbf{e}^T\mathbf{x}(t) = 1 \tag{2.79}$$

for all values of t *where* $\mathbf{x}(t)$ *is defined.*

Proof. From Lemma 2.11 we have

$$A\frac{d\mathbf{x}}{dt} = A\mathbf{f}(\mathbf{x}) = \mathbf{0}, \tag{2.80}$$

$$\mathbf{e}^T\frac{d\mathbf{x}}{dt} = \mathbf{e}^T\mathbf{f}(\mathbf{x}) = 0, \tag{2.81}$$

so that the thesis follows immediately from the assumption (2.74), (2.75) on $\mathbf{x}_0$ and the fundamental theorem of calculus. ∎

For $\mathbf{x} \in R^n$ let $E \in R^{n\times n}$ be the matrix given by

$$E = \mathrm{Diag}\left(A^T(AXA^T)^{-1}AX\mathbf{c}\right), \qquad \mathbf{x} \in R^n, \tag{2.82}$$

and $C \in R^{n\times n}$ be the matrix $C = \mathrm{Diag}(\mathbf{c})$. Let $J(\mathbf{x}) \in R^{n\times n}$ be the following matrix:

$$J(\mathbf{x}) = -\left[I - XA^T(AXA^T)^{-1}A - X\mathbf{e}\mathbf{e}^T\right](C - E) + (\mathbf{e}^TX\mathbf{c})I, \qquad \mathbf{x} \in R^n, \tag{2.83}$$

where we use $\bar{A}$ instead of A, as in (2.10), if AXA^T is of rank less than m. Let $\mathscr{W}$ be the set (2.47). An elementary computation shows that for $\mathbf{x} \in \mathscr{W}$, $J(\mathbf{x})$ is the Jacobian matrix of $\mathbf{f}(\mathbf{x})$ with respect to $\mathbf{x}$. Moreover let $S(\mathbf{x}^*, \rho^*)$ be the open ball of Lemma 2.5. We observe that for $\mathbf{x} \in \mathring{\Gamma} \cup S(\mathbf{x}^*, \rho^*)$, since the matrix AXA^T is invertible, $J(\mathbf{x})$ is continuous for $\mathbf{x} \in \mathring{\Gamma} \cup S(\mathbf{x}^*, \rho^*)$.

THEOREM 2.13. *Let us assume that the linear programming problem* (1.1)–(1.4) *has a unique nondegenerate minimizer* $\mathbf{x}^*$, *and let* $J(\mathbf{x}^*)$ *be given by* (2.83). *Then* $J(\mathbf{x}^*)$ *is invertible as an operator restricted to the subspace* $\begin{bmatrix} A \\ \mathbf{e}^T \end{bmatrix}^{\perp}$. *That is,* $J(\mathbf{x}^*)\mathbf{v} \neq \mathbf{0}$ *for each* $\mathbf{v} \neq \mathbf{0}$ *such that*

$$\mathbf{v} \in \begin{bmatrix} A \\ \mathbf{e}^T \end{bmatrix}^{\perp}.$$

Proof. First of all we show that for $i = 1, 2, \ldots, n$ we have

$$(C-E)_{ii} = 0 \quad \text{if and only if} \quad x_i^* \neq 0.$$

Let $X^* = \mathrm{Diag}(x^*)$. From Theorem 2.7 for $\alpha = 1$ we have

$$X^{*1/2}(C-E) = \mathrm{Diag}\Big(\big(\Pi_{(AX^{*1/2})}\, z_1 \cdot (X^{*1/2}c)\big)\Big) = 0. \tag{2.84}$$

Since $C - E$ is a diagonal matrix, from (2.84) we have that $X_{ii}^* \neq 0$ implies $(C - E)_{ii} = 0$ for $i = 1, 2, \ldots, n$. Let us show that $(C - E)_{ii} = 0$ implies $X_{ii}^* \neq 0$ for $i = 1, 2, \ldots, n$. In fact if we assume that there exists h such that $(C - E)_{hh} = 0$ and $x_h^* = 0$, then from the assumption that x^* is a nondegenerate minimizer of the linear programming problem (1.1)–(1.4) it follows that there exists a pivot transformation that makes the hth component of x^* nonzero. Let y^* be the new basic feasible solution corresponding to x^* via the pivot transformation. Since only one pivot operation has been made, the nonbasic components other than the hth component are still nonbasic, that is, $y_i^* = 0$ for each $i \neq h$ such that $x_i^* = 0$. Let $Y^* = \mathrm{Diag}(y_1^*, y_2^*, \ldots, y_n^*)$. From (2.84) we have

$$Y^*(C - E) = 0. \tag{2.85}$$

Moreover, since $e^T Y^* e = 1$, from (2.85) we have

$$0 = e^T Y^*(C - E)e = e^T Y^* Ce - e^T Y^* A^T (AX^* A^T)^{-1} AX^* c = e^T Y^* c = c^T y^*. \tag{2.86}$$

Therefore y^* would be a new minimizer of the linear programming problem (1.1)–(1.4), different from x^*, and this is absurd.

We have $J(x^*)^- = (C - E)^-$. In fact it is obvious that $v \in (C - E)^\perp$ implies $v \in J(x^*)^-$. Moreover, let

$$M = \left[A^T (AX^* A^T)^{-1} A - ee^T\right](C - E). \tag{2.87}$$

Then

$$J(x^*) = (C - E) - X^* M, \tag{2.88}$$

so that $J(x^*)v = 0$ implies $(C - E)v = X^* Mv$. Since X^* is a diagonal matrix,

we obtain $(X^*Mv)_i = (X^*)_{ii}(Mv)_i$, $i = 1,2,\ldots,n$. We have two cases:

(i) $X_{ii}^* = 0$, which implies $((C - E)v)_i = 0$;
(ii) $X_{ii}^* \neq 0$, which implies $(C - E)_{ii} = 0$.

Summarizing, we have that $((C - E)v)_i = 0$, $i = 1,2,\ldots,n$, that is, $v \in (C - E)^{\perp}$.

Now let $u \in R^n$ be such that

$$Au = 0, \qquad e^T u = 0. \tag{2.89}$$

We assume that $u \in J(x^*)^-$; since $x^* \in (C - E)^-$, then $z = x^* + u \in J(x^*)^{\perp}$. Moreover,

$$Az = 0, \qquad e^T z = 1, \tag{2.90}$$

and $z \in J(x^*)^-$ implies $z \in (C - E)^{\perp}$. If $z \in (C - E)^{\perp}$, then $z_i = 0$ for each i such that $x_i^* = 0$; this condition, together with (2.90), is a characterization of the minimizer x^* of the linear programming problem (1.1)–(1.4). Therefore $u = 0$. This concludes the proof. ∎

THEOREM 2.14. *Let x^* be the unique nondegenerate minimizer of the linear programming problem with normalized objective function* (1.1)–(1.4), *and $f(x)$ be given by* (2.49). *We consider the initial value problem*

$$\frac{dx}{dt} = f(x), \tag{2.91}$$

$$x(0) = \frac{1}{n}e. \tag{2.92}$$

Then a solution $x(t, (1/n)e)$ of (2.91), (2.92) *exists for $t \in [0, \infty)$, and*

$$\lim_{t \to \infty} x\left(t, \frac{1}{n}e\right) = x^*. \tag{2.93}$$

Proof. The standard existence and uniqueness theorems for the initial value problem for ordinary differential equations guarantee that the solution of (2.91), (2.92) exists locally. From Lemma 2.9 it follows that $f(x)$ is

tangential to $\partial\Lambda$, so that from Lemma 2.12 and the fact that $(1/n)\mathbf{e} \in \mathring{\Lambda}$ we have that $\mathbf{x}(t,(1/n)\mathbf{e}) \in \Lambda$. Moreover for $\mathbf{x} \in \mathring{\Lambda}$ we have

$$\frac{d}{dt}\left[\mathbf{c}^T\mathbf{x}\left(t,\frac{1}{n}\mathbf{e}\right)\right] = \mathbf{c}^T\frac{d\mathbf{x}}{dt} = -\left\|\Pi_{\left[\frac{AX^{1/2}}{\mathbf{e}^TX^{1/2}}\right]}\cdot(X^{1/2}\mathbf{c})\right\|^2 < 0. \quad (2.94)$$

that is, the objective function $\mathbf{c}^T\mathbf{x}$ is monotonically decreasing along the trajectory $\mathbf{x}(t,(1/n)\mathbf{e})$. Since the minimum of $\mathbf{c}^T\mathbf{x}$ on Λ is zero, $\mathbf{x}^*$ is the unique minimizer of $\mathbf{c}^T\mathbf{x}$ on Λ, and $\bar{f}(\mathbf{x}^*) = 0$, from G. Sansone and R. Conti [21, p. 31] we have that $\mathbf{x}^*$ is the unique limit point of $\mathbf{x}(t,(1/n)\mathbf{e})$ and

$$\lim_{t\to\infty}\mathbf{x}\left(t,\frac{1}{n}\mathbf{e}\right) = \mathbf{x}^*. \quad (2.95)$$

This concludes the proof. ■

3. THE QUADRATIC ALGORITHM FOR LINEAR PROGRAMMING

Let $\mathbf{x} \in R^n$, $D \subset R^n$ be an open set, and $\bar{D}$ be the closure of D; let $\mathbf{w}: \bar{D} \subseteq R^n \to R^n$ be a function continuously differentiable in D, whose Jacobian matrix is denoted by $Q(\mathbf{x}) = \partial\mathbf{w}/\partial\mathbf{x}$. Let us consider the initial value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{w}(\mathbf{x}), \quad (3.1)$$

$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}_0 \in D. \quad (3.2)$$

Let I be the $n \times n$ identity matrix, $h_k > 0$, $k = 0,1,2,\ldots$ be a sequence of stepsizes, and $t_k = \sum_{i=0}^{k} h_i$. Then any solution $\mathbf{x}(t_k)$ of (3.1), (3.2) can be approximated with $\mathbf{x}^k$ computed as follows:

$$\mathbf{x}^0 = \mathbf{x}_0, \quad (3.3)$$

$$\left[I - h_kQ(\mathbf{x}^k)\right]\mathbf{s}^k = h_k\mathbf{w}(\mathbf{x}^k), \qquad k = 0,1,2,\ldots, \quad (3.4)$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{s}^k, \qquad k = 0,1,2,\ldots. \quad (3.5)$$

The numerical scheme (3.3)–(3.5) to integrate the initial value problem (3.1), (3.2) is A-stable and linearly implicit, and has been studied by J. D. Lambert and S. T. Sigurdsson in [22].

Let $\mathbf{f}(\mathbf{x})$ be the vector field given by (2.49), and $J(\mathbf{x})$ be its Jacobian matrix given by (2.83). We will apply the numerical scheme (3.3)–(3.5) to the initial value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}). \tag{3.6}$$

$$\mathbf{x}(0) = \frac{1}{n}\mathbf{e} \tag{3.7}$$

considered in Section 2. Let $\mathring{\Gamma}$ be given by (2.9). By Lemma 2.5 there exists a neighborhood $S(\mathbf{x}^*, \rho^*)$ of $\mathbf{x}^*$ such that $\mathbf{f}(\mathbf{x})$ is a continuously differentiable function in $\mathring{\Gamma} \cup S(\mathbf{x}^*, \rho^*)$.

LEMMA 3.1. *Let $\mathbf{x}^*$ be the unique minimizer of the linear programming problem with normalized objective function (1.1)–(1.4). Let us apply the numerical scheme (3.3)–(3.5) to the initial value problem (3.6), (3.7). Moreover let*

$$h_k \neq (\mathbf{e}^T X_k \mathbf{c})^{-1} \qquad \text{for} \quad k = 0, 1, 2, \ldots \tag{3.8}$$

$$I - h_k J(\mathbf{x}^k) \text{ be invertible} \qquad \text{for} \quad k = 0, 1, 2, \ldots \tag{3.9}$$

Then the sequence $\{\mathbf{x}^{k+1}\}$, $k = 0, 1, 2, \ldots$, generated by (3.3)–(3.5) exists, and $\mathbf{s}^k$ satisfies

$$A\mathbf{s}^k = \mathbf{0}, \qquad k = 0, 1, 2, \ldots \tag{3.10}$$

$$\mathbf{e}^T\mathbf{s}^k = 0, \qquad k = 0, 1, 2, \ldots \tag{3.11}$$

Proof. We note that $\mathbf{x}^0 = (1/n)\mathbf{e}$ is a feasible point of the linear programming problem (1.1)–(1.4) and that $\mathbf{s}^k$ is defined by

$$\left[I - h_k J(\mathbf{x}^k)\right]\mathbf{s}^k = h_k \mathbf{f}(\mathbf{x}^k), \qquad k = 0, 1, 2, \ldots \tag{3.12}$$

In Lemma 2.11 it has been shown that if $\mathbf{x}^k$ is a feasible point for the linear

programming problem (1.1)–(1.4), we have

$$Af(x^k) = 0, \tag{3.13}$$

$$e^T f(x^k) = 0, \tag{3.14}$$

so that applying A to both sides of (3.12), we obtain

$$A\left[I - h_k J(x^k)\right]s^k = 0. \tag{3.15}$$

From (2.83) we have

$$AJ(x^k) = (e^T X_k c)A. \tag{3.16}$$

Therefore we have

$$\left[1 - h_k(e^T X_k c)\right]As^k = 0, \tag{3.17}$$

so that from (3.8) we have $As^k = 0$, $k = 0, 1, 2, \ldots$.

Moreover it is easy to verify that

$$e^T J(x^k) = (e^T X_k c)e^T I, \tag{3.18}$$

so that from Lemma 2.11 we have

$$e^T\left[I - h_k J(x^k)\right]s^k = 0, \tag{3.19}$$

which implies

$$\left[1 - h_k(e^T X_k c)\right]e^T s^k = 0, \tag{3.20}$$

so that from (3.8) we have $e^T s^k = 0$, $k = 0, 1, 2, \ldots$. ■

Let $D \subseteq R^n$ be an open set and $D_0 \subset D$ be a convex set.

DEFINITION 3.2. Let $w(x): D \subseteq R^n \to R^n$ be a continuously differentiable function. Let $\xi \in R^m$, D_ξ be an open neighborhood of ξ, and $T: D \times D_\xi \subset R^n \times R^m \to L(R^n)$, where $L(R^n)$ is the set of the $n \times n$ matrices. Then $T(x, \xi)$ is a consistent approximation to the Jacobian matrix $Q(\xi)$ of $w(x)$ on

$D_0 \subset D$ if $0 \in R^m$ is a limit point of D_ξ and

$$\lim_{\substack{\xi \to 0 \\ \xi \in D_\xi}} T(x, \xi) = Q(x) \tag{3.21}$$

uniformly for $x \in D_0$. Moreover, if there exist two constants $c > 0$ and $r > 0$ such that

$$\| Q(x) - T(x, \xi) \| \leq c \|\xi\| \tag{3.22}$$

for each $x \in D_0$ and $\xi \in D_\xi \cap S(0, r)$, where $S(0, r) = \{\xi \in R^m \mid \|\xi\| < r\}$, then $T(x, \xi)$ is a strongly consistent approximation to $Q(x)$ on D_0.

LEMMA 3.3. *Let $D \subseteq R^n$ be an open set, and $w: D \subseteq R^n \to R^n$ be a continuously differentiable function on the convex set $D_0 \subset D$. Let $Q(x) \in \mathrm{Lip}_\gamma(D_0)$, that is, let $Q(x)$ be a Lipschitz continuous function for $x \in D_0$ with Lipschitz constant $\gamma \geq 0$. Then*

$$\| w(y) - w(x) - Q(x)(y - x) \| \leq \frac{\gamma}{2} \|x - y\|^2 \tag{3.23}$$

for each $x, y \in D_0$.

Proof. See J. M. Ortega and W. C. Rheinboldt [20, Theorem 3.2.12, p. 73]. ∎

THEOREM 3.4. *Let x^* be the unique nondegenerate minimizer of the linear programming problem with normalized objective function (1.1)–(1.4), and let*

$$B = \begin{bmatrix} A \\ e^T \end{bmatrix}$$

and B^- be given by (2.8). Moreover, let $f(x)$ be given by (2.49), and

$$h_k = \frac{\alpha_k}{\| f(x^*) \|} \qquad k = 0, 1, 2, \ldots \tag{3.24}$$

where $\alpha_k > a > 0$ is a bounded sequence such that $h_k \neq (e^T X_k c)^{-1}$. Then

there exists $\rho_1 > 0$ *such that open neighborhood* $S(x^*, \rho_1) = \{x \in R^n \mid \|x - x^*\| < \rho_1\}$ *of* x^* *such that* $x^0 \in S(x^*, \rho_1) \cap \Sigma$, *where* Σ *is given by* (2.39), *then the sequence* $\{x^k\}$, $k =$ [illegible]... *generated by*

$$x^0 = \zeta_0, \tag{3.25}$$

$$[I - h_k J(x^{k-1})](x^{k-1} - x^k) = h_k f(x^k), \qquad k = 0, 1, 2, \ldots, \tag{3.26}$$

where $J(x)$ *is given by* (2.33) *and the linear system* (3.26) *is solved in* B^-, *is well defined,* $x^{k-1} \in S(x^*, \rho_1) \cap \Sigma$ *for* $k = 0, 1, 2, \ldots$ *and* $\{x^k\}$, $k = 0, 1, 2, \ldots$, *converges quadratically to* x^*.

Proof. We argue by induction on the index k. Let

$$\xi_k = \frac{1}{h_k}, \qquad k = 0, 1, 2, \ldots \tag{3.27}$$

and

$$\Phi(x^k, \xi_k) = -\xi_k I + J(x^k), \qquad k = 0, 1, 2, \ldots. \tag{3.28}$$

We observe that (3.26) can be rewritten as follows:

$$-\Phi(x^k, \xi_k)(x^{k-1} - x^k) = f(x^k), \qquad k = 0, 1, 2, \ldots. \tag{3.29}$$

It is easy to see that $\Phi(x, \xi)$ is a strongly consistent approximation of $J(x)$ on Γ when ξ goes to ∞. We have seen in Lemma 3.1 that if $v \in R^n$ and $x^k \in \Sigma$ we have

$$\Phi(x^k, \xi_k)v = (e^T X_k c - \xi_k)Av, \tag{3.30}$$

$$\Phi(x^k, \xi_k)v = (e^T X_k c - \xi_k)e^T v. \tag{3.31}$$

From (3.24) and (3.25) it follows that $\xi_k \neq e^T X_k c$ for $k = 0, 1, 2, \ldots$, so that we have

$$\Phi(x^k, \xi_k)v \in B^- \quad \text{if and only if} \quad v \in B^-. \tag{3.32}$$

From Theorem 2.13 we have that $J(x^*)$ restricted to $B^{\perp}$ is invertible and

$$\left\| [J(x^*)]|_{B^{\perp}}^{-1} [J(x^*) - \Phi(x^k, \xi_k)]|_{B^{\perp}} \right\| < 1 \tag{3.33}$$

for $x^k \in S(x^*, \rho_1)$ in a suitable neighborhood of x^* and ξ_k in a suitable neighborhood U of zero. The perturbation lemma (see J. M. Ortega and W. C. Rheinboldt [20, Lemma 2.3.2, p. 45]) implies that the inverse of the linear operator $\Phi(x^k, \xi_k)$ restricted to the subspace $B^{\perp}$ exists when $x^k \in S(x^*, \rho_1)$, $\xi_k \in U$. From Lemma 2.11 we have that $x^k \in \Sigma$ implies $f(x^k) \in B^{\perp}$. When $x^k \in S(x^*, \rho_1) \cap \Sigma$ and $\xi_k \in U$, from the fact that $f(x^k) \in B^{\perp}$ and (3.32), (3.33) it follows that x^{k+1} is well defined and $x^{k+1} \in \Sigma$. Moreover there exists $\eta > 0$ such that

$$\left\| \Phi(x, \xi)|_{B^{\perp}}^{-1} \right\| \leq \eta, \qquad x \in S(x^*, \rho_1), \quad \xi \in U, \tag{3.34}$$

and we have

$$\begin{aligned} \|x^{k+1} - x^*\| &= \left\| \Phi(x^k, \xi_k)|_{B^{\perp}}^{-1} [\Phi(x^k, \xi_k)(x^k - x^*) - f(x^k)] \right\| \\ &\leq \eta \left(\|\Phi(x^k, \xi_k) - J(x^k)\| + \|J(x^k) - J(x^*)\| \right) \|x^k - x^*\| \\ &\quad + \eta \|f(x^*) - f(x^k) - J(x^*)(x^k - x^*)\|. \end{aligned} \tag{3.35}$$

Since we can always choose $\rho_1 > 0$ such that $J(x) \in \mathrm{Lip}_{\gamma}(S(x^*, \rho_1))$ for some $\gamma > 0$, from Lemma 3.3 we have

$$\begin{aligned} \|x^{k+1} - x^*\| &\leq \xi_k \eta \|x^k - x^*\| + \eta \gamma \|x^k - x^*\|^2 + \frac{\eta \gamma}{2} \|x^k - x^*\|^2 \\ &\leq \omega(x^k, \xi_k) \|x^k - x^*\|, \end{aligned} \tag{3.36}$$

where

$$\omega(x^k, \xi_k) = \eta \xi_k + \tfrac{1}{2} \eta \gamma \|x^k - x^*\|. \tag{3.37}$$

The neighborhoods $S(x^*, \rho_1)$ and U can be chosen in such a way that

$$\omega(x^k, \xi_k) \leq \varkappa < 1, \qquad x^k \in S(x^k, \rho_1) \cap \Sigma, \quad \xi_k \in U. \tag{3.38}$$

From (3.36) and (3.38) we have

$$\|x^{k+1} - x^*\| \leq \varkappa \|x^k - x^*\|, \qquad x^k \in S(x^*, \rho_1) \cap \Sigma. \tag{3.39}$$

Therefore $x^{k+1} \in S(x^*, \rho_1) \cap \Sigma$. Moreover we have

$$\|x^{k+1} - x^*\| \leq \varkappa^{k+1} \|x^0 - x^*\|, \tag{3.40}$$

so that the iterates $\{x^k\}$, $k = 0, 1, 2, \ldots$ given by (3.26) are well defined if $x^0 \in S(x^*, \rho_1) \cap \Sigma$ and

$$\lim_{k \to \infty} x^k = x^*. \tag{3.41}$$

Moreover, since $f(x)$ is a continuously differentiable function in a neighborhood of x^* and $f(x^*) = 0$, there exists a constant $M > 0$ such that

$$\|f(x^k)\| \leq M \|x^k - x^*\|, \qquad x^k \in S(x^*, \rho_1) \cap \Sigma. \tag{3.42}$$

From (3.24) we obtain

$$\xi_k \leq \frac{\|f(x^k)\|}{\alpha_k} \leq \frac{M}{\alpha_k} \|x^k - x^*\|, \qquad x^k \in S(x^*, \rho_1) \cap \Sigma. \tag{3.43}$$

That is, the sequence $\{x^k\}$ converges quadratically to x^*. This concludes the proof. ■

4. NUMERICAL EXPERIMENTS

We begin by comparing the computational cost of a step of the algorithm introduced in Section 3 with that of a step of the simplex algorithm or of a step of Karmarkar's algorithm.

We consider the linear programming problem in the canonical form (1.1)–(1.4). It is easy to verify that the computational cost of a step of the simplex algorithm is given by

$$mn + \text{lower order terms}. \tag{4.1}$$

The computational cost of one step of Karmarkar's algorithm is essentially due to the computation of the matrix

$$AX^2A^T \tag{4.2}$$

and to the solution of the $m \times m$ linear system

$$(AX^2A^T)y = AX^2c. \tag{4.3}$$

Since the matrix AX^2A^T is symmetric, its computation requires

$$\frac{m^2n}{2} + \text{lower order terms} \tag{4.4}$$

elementary operations while the solution of the linear system (4.3) requires

$$\frac{m^3}{6} + \text{lower order terms} \tag{4.5}$$

elementary operations. Since $m < n$, we can conclude that the computational cost of one step of Karmarkar's algorithm is roughly

$$\tfrac{7}{6}n^3 + \text{lower order terms} \tag{4.6}$$

elementary operations. This cost can be reduced using some special procedures: for example in [2] Karmarkar has shown that the use of successive rank-one modifications to compute (4.2), (4.3) reduces the "average" computational cost of each step to

$$cn^{2.5} + \text{lower order terms} \tag{4.7}$$

elementary operations for some constant $c > 0$.

The computational cost of one step of the algorithm introduced in Section 3 is essentially due to the computation of the matrix

$$AXA^T. \tag{4.8}$$

to the solution of the linear system

$$(A\Sigma A^T)y = A\Sigma c, \tag{4.9}$$

to the computation of the matrix

$$A^T(A\Sigma A^T)^{-1}A \tag{4.10}$$

that appears in the expression of the Jacobian $J(x)$ (2.53), and to the solution of the linear system (3.26). The computational costs of (4.8), (4.9) are analogous to those of (4.2), (4.3) respectively. Moreover the computational cost of the solution of the linear system (3.26) is

$$\frac{n^3}{3} + \text{lower order terms} \tag{4.11}$$

elementary operations.

In order to compute (4.10) we use the Cholesky decomposition of $A\Sigma A^T$, that is,

$$A\Sigma A^T = LL^T, \tag{4.12}$$

where $L \in R^{m \times m}$ is a nonsingular lower triangular matrix. So we have

$$(A\Sigma A^T)^{-1} = (L^{-1})^T L^{-1} \tag{4.13}$$

and

$$A^T(A\Sigma A^T)^{-1}A = (L^{-1}A)^T(L^{-1}A). \tag{4.14}$$

Since in order to compute $L^{-1}A$

$$\frac{m^3}{6} + \frac{m^2 n}{2} + \text{lower order terms} \tag{4.15}$$

elementary operations are necessary, and the matrix $(L^{-1}A)^T(L^{-1}A)$ is

symmetric, the computational cost of (4.10) is

$$\frac{m^2 n}{2} + \text{lower order terms} \tag{4.16}$$

elementary operations. Since $m < n$, we can conclude that the computational cost of one step of the algorithm introduced in Section 3 is

$$\tfrac{13}{6} n^3 + \text{lower order terms} \tag{4.17}$$

elementary operations. Moreover, if we use successive rank-one modifications, as proposed in [2], we can decrease the "average" computational cost of each step to

$$c' n^{2.5} + \text{lower order terms} \tag{4.18}$$

elementary operations, for some constant $c' > 0$. Moreover, to improve the value of c' it is possible to use any combination of the ideas proposed in [7, 23, 24].

To conclude, the computational cost of one step of the algorithm introduced in Section 3 is of the same order as that of one step of Karmarkar's algorithm, while one step of the simplex algorithm is much cheaper. However, due to the quadratic convergence of our algorithm, we expect that the number of iterations needed to solve a linear programming problem to a given accuracy should be approximately independent of the problem size n.

We present now some numerical results that support our expectation. The algorithm described in Section 3 has been implemented using two special expedients to avoid failure due to the ill-conditioning of the problem considered.

The matrix $A \in R^{m \times n}$ given by (1.2) is replaced with the matrix $\tilde{A} \in R^{m \times n}$ to reduce its condition number. $\tilde{A}$ is obtained using the singular value decomposition of AA^T. This decomposition has a computational cost of order n^3, so it costs the same as one step of the algorithm described in Section 3. Let $AA^T = Q^T D V$ be the singular value decomposition of AA^T; then the matrix $\tilde{A}$ is given by

$$\tilde{A} = (Q^T D^* V) A, \tag{4.19}$$

where $D^* \in R^{m \times m}$ is a diagonal matrix such that $(D^*)_{ii} = 1/D_{ii}$ if $D_{ii} > 0$ and $(D^*)_{ii} = 1$ if $D_{ii} = 0$ for $i = 1, \ldots, m$.

In the first k_1 steps of our algorithm ($k_1 \leq 5$ in our numerical experiments), the Riemannian metric

$$G(x) = X^{-1} = \mathrm{Diag}(x^{-1}) \tag{4.20}$$

is replaced with

$$\tilde{G}(x) = M_k X^{-1}, \tag{4.21}$$

where $M_k = X_k^{-1} = \mathrm{Diag}(x_k^{-1})$ and x_k is the current point at step k. We note that in order to apply our algorithm is not necessary to have a normalized objective function—that is, is not necessary to know the value of the objective function at the minimizer. However, in our numerical experiments we use test problems with normalized objective function. The stopping rule used is

$$\nu_k = c^T x^k \leq 1.0 \times 10^{-8} \left(c^T \frac{e}{n} \right). \tag{4.22}$$

We have considered ten test problems. Problem 1 (ZIR1) is a problem coming from the operation of an industrial plant in central Italy. The other problems come from the System Optimization Laboratory at Stanford University and have been made available to us through NETLIB [25]. The numbers of variables (n) and of constraints (m), shown in Table 1, are those relative to the problems in canonical form. Finally, k denotes the index of the first step that verifies the stopping rule (4.22).

We note that in Table 1, while n, m vary by an order of magnitude, the number k of steps needed to solve the problem varies only by a factor of two. Moreover, test problems with $n, m \leq 5$ are solved in about ten steps.

TABLE 1

Test problem		m	n	k	ν_k
1.	ZIR1	304	543	21	2.41D-10
2.	ADLITTLE	57	141	21	3.16D-09
3	AFIRO	28	54	12	1.52D-12
4.	BEACONFD	173	295	20	3.91D-09
5.	BLEND	75	117	21	1.47D-12
6.	ISRAEL	175	319	17	1.94D-10
7.	SC105	106	166	13	1.36D-11
8.	SC50A	51	51	14	1.48D-14
9.	SC50B	51	81	11	7.84D-10
10.	SHARE2B	97	167	21	1.78D-10

The algorithm has been coded in FORTRAN and tested on a VAX/VMS Version V5.1 in double precision arithmetic.

REFERENCES

1 N. Hooker, Karmarkar's linear programming algorithm, *Interfaces* 16:75–90 (1986).

2 N. Karmarkar, A new polynomial time algorithm in linear programming, *Combinatorica* 4:373–395 (1984).

3 G. B. Dantzig, *Linear Programming and Extensions*, Princeton U.P., Princeton, N.J., 1963.

4 V. Klee and G. L. Minty, How good is the simplex algorithm?, in *Inequalities III* (O. Shisha, Ed.), Academic, New York, 1972, pp. 159–179.

5 S. Smale, On the average number of steps in the simplex method of linear programming, *Math. Programming* 27:241–262 (1983).

6 L. G. Khachiyan, A polynomial algorithm in linear programming, *Dokl. Akad. Nauk SSSR N. S.* 244:1093–1096 (1979); English trans., *Soviet Math. Dokl.* 20:191–194 (1979).

7 P. E. Gill, W. Murray, M. A. Saunders, J. A. Tomlin, and M. H. Wright, On projected Newton barrier methods for linear programming and an equivalence to Karmarkar's projective method, *Math. Programming* 36:183–209 (1986).

8 J. Renegar, A polynomial-time algorithm based on Newton's method for linear programming, *Math. Programming* 40:50–94 (1988).

9 C. Gonzaga, An algorithm for solving linear programming problems in $O(n^3L)$ operations, in *Progress in Mathematical Programming* (N. Megiddo, Ed.), Springer-Verlag, New York, 1989, pp. 1–28.

10 M. Iri and H. Imai, A multiplicative barrier function method for linear programming, *Algorithmica* 1:455–482 (1986).

11 N. Megiddo, Pathways to the optimal set in linear programming, in *Progress in Mathematical Programming* (N. Megiddo, Ed.), Springer-Verlag, New York, 1989, pp. 151–158.

12 J. A. Tomlin, A note on comparing simplex and interior methods for linear programming, in *Progress in Mathematical Programming* (N. Megiddo, Ed.), Springer-Verlag, New York, 1989, pp. 91–105.

13 D. A. Bayer and J. C. Lagarias, The nonlinear geometry of linear programming I, *Trans. Amer. Math. Soc.* 314:499–526 (1989).

14 F. Zirilli, The use of ordinary differential equations in the solution of nonlinear systems of equations, in *Nonlinear Optimization 1981* (M. T. D. Powell, Ed.), Academic, New York, 1982, pp. 39–47.

15 F. Zirilli, The solution of nonlinear systems of equations by second order systems of o.d.e. and linearly implicit A-stable techniques, *SIAM J. Numer. Anal.* 19:800–816 (1982).

16 F. Aluffi-Pentini, V. Parisi, and F. Zirilli, A differential equation algorithm for nonlinear equations, *ACM Trans. Math. Software* 10:299–316 (1984).

17 F. Aluffi-Pentini, V. Parisi, and F. Zirilli, Algorithm 617 DAFNE: A differential equation algorithm for nonlinear equations, *ACM Trans. Math. Software* 10:317–324 (1984).

18 J. L. Nazareth, Homotopy techniques in linear programming, *Algorithmica* 1:529–535 (1986).

19 N. Karmarkar, J. C. Lagarias, L. Slutsman, and P. Wang, Power series variants of Karmarkar-type algorithms, *AT&T Tech. J.*, to appear.

20 J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations*, McGraw-Hill, New York, 1953.

21 G. Sansone and R. Conti, *Nonlinear Differential Equations*, Pergamon, New York, 1974.

22 J. D. Lambert and S. T. Sigurdsson, Multistep methods with variable matrix coefficients, *SIAM J. Numer. Anal.* 9:715–733 (1972).

23 D. F. Shanno, Computing Karmarkar projections quickly, *Math. Programming* 41:61–71 (1988).

24 J. E. Dennis, A. M. Morshedi, and K. Turner, A variable-metric variant of the Karmarkar algorithm for linear programming, *Math. Programming* 39:1–20 (1987).

25 D. M. Gay, Electronic mail distribution of linear programming test problems, *Committee on Algorithms Newsletter*, *Math. Programming Soc.* 13 (1985).

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